Exact discretization, tight frames, and recovery via D-optimal designs

Felix Bartel   Lutz Kämmerer   Kateryna Pozharska

Martin Schäfer   Tino Ullrich

MCM 2025

31th of July 2025

UNSW SYDNEY

L_2 Marcinkiewicz-Zygmund (MZ) inequalities

Definition [Marcinkiewicz, Zygmund 1937]

Let
- D be a domain,
- μ a measure on D,
- $L_2 = L_2(D, \mu)$ the space of square-integrable functions, and
- V an m-dimensional function space.

Points $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n \in D$ and weights $\omega_1, \ldots, \omega_n > 0$ fulfill an **L_2 MZ inequality** with constants $0 < (1-\varepsilon) \le (1+\varepsilon) < \infty$, iff

$$(1-\varepsilon)\|f\|_{L_2}^2 \le \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \le (1+\varepsilon)\|f\|_{L_2}^2 \quad \text{for all} \quad f \in V\,.$$

L_2 Marcinkiewicz-Zygmund (MZ) inequalities

Some existing work: [Mhaskar, Narcowich, Ward 2001], [Nuyens, Cools 2006], [Keiner, Kunis, Potts 2007], [Filbir, Mhaskar 2011], [Müller-Gronbach, Novak, Ritter 2012], [Kämmerer, Potts, Volkmer 2015], [Temlyakov 2018], [Trefethen 2019], [Gröchenig 2020], [Filbir, Hielscher, Jahn, T. Ullrich 2024], …

Goal: Construct L_2 MZ inequalities with

- small number of points n compared to $\dim(V) = m$
- close constants, ideally $\varepsilon = 0$

Random points

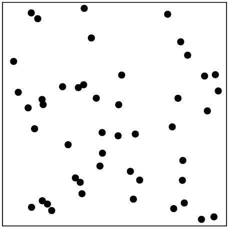
Theorem (random points)

[Tropp 2010][Cohen, Davenport, Leviatan 2017]

Let
- V function space with ONB $\{\eta_1, \dots, \eta_m\}$,
- $\varepsilon > 0$, $n \geq (6m\log m)/\varepsilon^2$,
- $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n$ i.i.d. w.r.t. $\frac{1}{m}\sum_{k=1}^m |\eta_k(\boldsymbol{x})|^2 \, \mathrm{d}\mu$,
- $\omega_i = m(\sum_{k=1}^m |\eta_k(\boldsymbol{x}^i)|^2)^{-1}$.

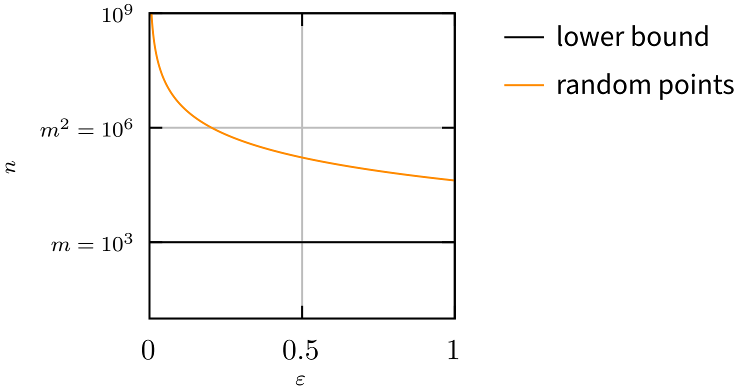
Then with high probability it holds for all $f \in V$

$$(1-\varepsilon)\|f\|_{L_2}^2 \leq \sum_{i=1}^n \omega_i |f(\boldsymbol{x}^i)|^2 \leq (1+\varepsilon)\|f\|_{L_2}^2 \,.$$



random points

Random points



number of points for L_2 MZ inequality with constants $1-\varepsilon$ and $1+\varepsilon$
for $\dim(V) = m = 1000$

Subsampling

non-constructive subsampling: [Krieg, M. Ullrich 2021]

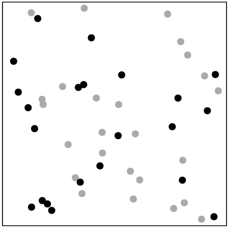
Theorem (subsampled points) [B., Schäfer, T. Ullrich 2023]

Let
- V function space with ONB $\{\eta_1, \dots, \eta_m\}$,
- $\varepsilon > 0$, $n = \left\lceil m \, \frac{1+\sqrt{1-\varepsilon^2}}{1-\sqrt{1-\varepsilon^2}} \right\rceil$.

Then there exist $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n$ and $\omega_1, \dots, \omega_n > 0$ with

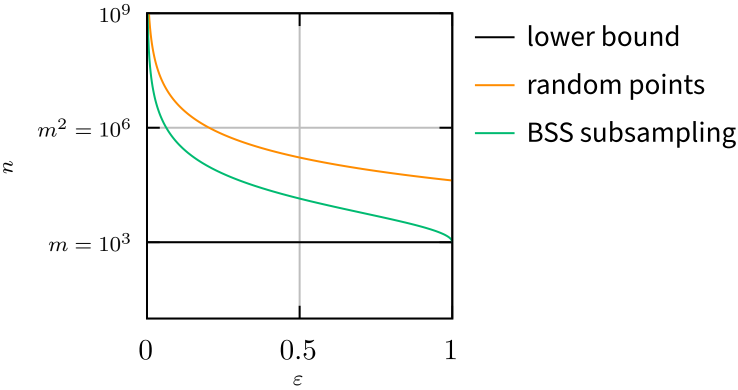
$$(1-\varepsilon)\|f\|_{L_2}^2 \le \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \le (1+\varepsilon)\|f\|_{L_2}^2 \,.$$

for all $f \in V$.



subsampled points

BSS subsampling



number of points for L_2 MZ inequality with constants $1-\varepsilon$ and $1+\varepsilon$
for $\dim(V) = m = 1000$

Q: How to construct general exact L_2 MZ inequalities?

Exact L_2 MZ inequalities

Theorem (connection to matrices)

Let
- V function space with ONB $\{\eta_1, \dots, \eta_m\}$,
- $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in D$, $\omega_1, \dots, \omega_n > 0$, and
- $\boldsymbol{F}(\boldsymbol{x}) = [\eta_k(\boldsymbol{x})]_{k=1,\dots,m} \in \mathbb{C}^m$.

Then

$$(1-\varepsilon)\|f\|_{L_2}^2 \le \sum_{i=1}^n \omega_i |f(\boldsymbol{x}^i)|^2 \le (1+\varepsilon)\|f\|_{L_2}^2 \quad \text{for all} \quad f \in V$$

iff

$$\Big\| \sum_{i=1}^n \omega_i \boldsymbol{F}(\boldsymbol{x}^i)\boldsymbol{F}(\boldsymbol{x}^i)^* - \boldsymbol{I} \Big\|_{2\to 2} \le \varepsilon\,.$$

Exact L_2 MZ inequalities

Problem of finding exact L_2 MZ inequalities in matrix language

Define

$$\mathcal{H} := \Big\{ \boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})^* : \boldsymbol{x} \in D \Big\} \subset \mathbb{C}^{m\times m} .$$

Then the set of L_2 MZ inequalities with weights adding up to one correspond to

$$\operatorname{conv} \mathcal{H} = \Big\{ \sum_{i=1}^{n} \omega_i \boldsymbol{F}(\boldsymbol{x}^i)\boldsymbol{F}(\boldsymbol{x}^i)^* : \boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in D, \omega_1 + \dots + \omega_n = 1, n \in \mathbb{N} \Big\} .$$

Goal: find zero of

$$\gamma\colon \operatorname{conv} \mathcal{H} \to [0,\infty),\ \boldsymbol{H} \mapsto \|\boldsymbol{H} - \boldsymbol{I}\|_{2\to 2} .$$

Exact L_2 MZ inequalities

Idea:

1. random L_2 MZ inequalities give

$$\forall \varepsilon > 0 \; \exists \boldsymbol{H}_\varepsilon \in \operatorname{conv} \mathcal{H} : \gamma(\boldsymbol{H}_\varepsilon) \le \varepsilon$$

2. continuity of γ and compactness of $\operatorname{conv} \mathcal{H}$ give

$$\exists \boldsymbol{H}^\star \in \operatorname{conv} \mathcal{H} : \gamma(\boldsymbol{H}^\star) = 0$$

3. Carathéodory gives

$$\boldsymbol{H}^\star = \sum_{i=1}^{m^2+1} \omega_i \boldsymbol{F}(\boldsymbol{x}^i) \boldsymbol{F}(\boldsymbol{x}^i)^*$$

Exact L_2 MZ inequalities

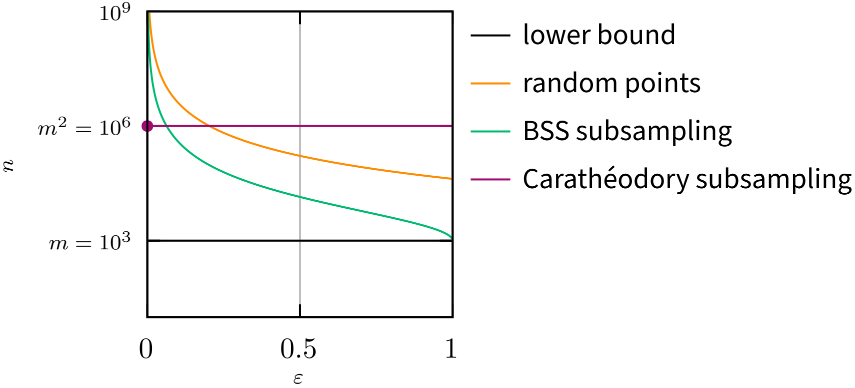
Theorem (exact MZ) [B., Kämmerer, Pozharska, Schäfer, T. Ullrich 2024]

Let
- D be a compact topological space,
- μ a probability measure,
- $V \subset C(D)$ with $\dim(V) = m$, and
- $n = \dim(\mathrm{span}\{f \cdot \overline{g} : f, g \in V\}) + 1 \leq m^2 + 1\,.$

Then there exist $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n \in D$ and $\omega_1 + \cdots + \omega_n = 1$ such that

$$\|f\|_{L_2}^2 = \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \quad \text{for all} \quad f \in V\,.$$

Carathéodory subsampling



number of points for L_2 MZ inequality with constants $1-\varepsilon$ and $1+\varepsilon$ for $\dim(V) = m = 1000$

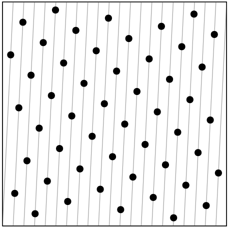
Exact L_2 MZ inequalities

Theorem (rank-1 lattices) [Kämmerer, Potts, Volkmer 2015]

Let
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$,
- $V = \operatorname{span}\{\exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with
- $I \subset [-\frac{|I|}{2}, \frac{|I|}{2}]^d \subset \mathbb{Z}^d$ symmetric.

Then for some $|I| \le n \le |I|^2$, there exists a rank-1 lattice $\boldsymbol{X} = \{\frac{i}{n}\boldsymbol{z} \bmod \boldsymbol{1}, i = 0, \dots, n-1\}$ such that

$$\|f\|_{L_2}^2 = \frac{1}{n}\sum_{\boldsymbol{x}\in\boldsymbol{X}} |f(\boldsymbol{x})|^2 \quad \text{for all} \quad f \in V\,.$$



$\boldsymbol{z} = (1, 21)^\top, n = 55$

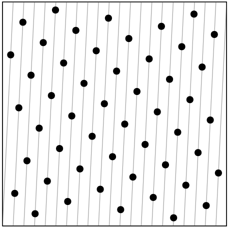
Exact L_2 MZ inequalities

Theorem (rank-1 lattices) [Kämmerer, Potts, Volkmer 2015]

Let
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$,
- $V = \operatorname{span}\{\exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with
- $I \subset [-\frac{|I|}{2}, \frac{|I|}{2}]^d \subset \mathbb{Z}^d$ symmetric.

Then for some $|I| \le n \le |I|^2$, there exists a rank-1 lattice $\boldsymbol{X} = \{\frac{i}{n}\boldsymbol{z} \bmod \boldsymbol{1}, i = 0, \dots, n-1\}$ such that

$$\|f\|_{L_2}^2 = \frac{1}{n} \sum_{\boldsymbol{x}\in\boldsymbol{X}} |f(\boldsymbol{x})|^2 \quad \text{for all} \quad f \in V\,.$$



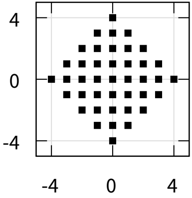
$\boldsymbol{z} = (1, 21)^\top, n = 55$

[B., Kämmerer, Pozharska, Schäfer, T. Ullrich 2024]: There exist points $\boldsymbol{x}^1, \dots, \boldsymbol{x}^{|I|^2} \in \mathbb{T}^d$ and weights $\omega_1 + \dots + \omega_{|I|^2} = 1$ such that

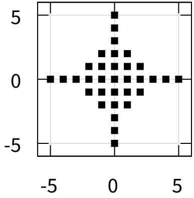
$$\|f\|_{L_2}^2 = \sum_{i=1}^{|I|^2} \omega_i |f(\boldsymbol{x}^i)|^2 \quad \text{for all} \quad f \in V\,.$$

Numerical experiments in $d = 2$

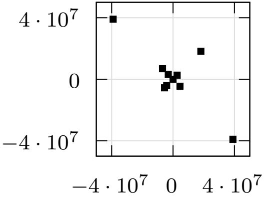
ℓ_1-ball $(m = 41)$



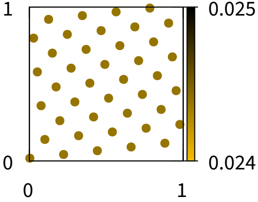
hyperbolic cross $(m = 33)$



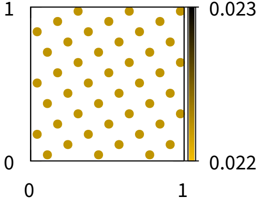
bad frequencies $(m = 10)$



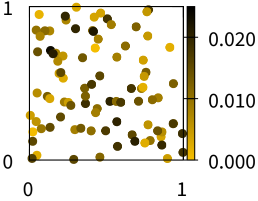
$n = 41, \varepsilon = 1.60834 \cdot 10^{-14}$



$n = 45, \varepsilon = 1.08850 \cdot 10^{-14}$



$n = 91, \varepsilon = 3.62941 \cdot 10^{-16}$



with $\varepsilon = \| \sum_{i=1}^{N+1} \omega_i \boldsymbol{\varphi}(\boldsymbol{x}^i) \cdot \boldsymbol{\varphi}(\boldsymbol{x}^i)^* - \boldsymbol{I} \|_{2 \to 2}$

Numerical experiments: bad frequencies

Setting:

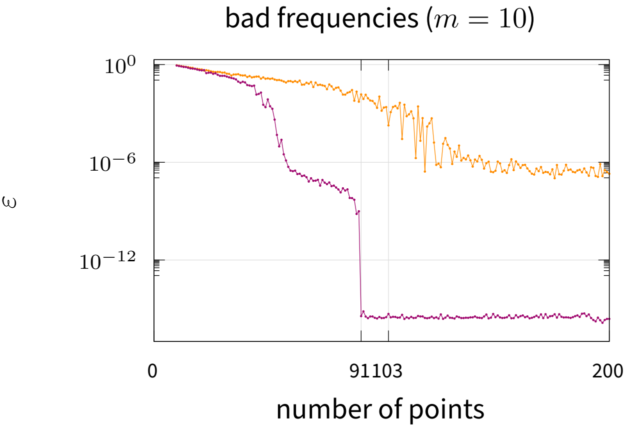
- dimension $d = 1$
- 10 frequencies

$$I = \Big\{0, 107062, 124928, 1033760, 1414818, 2142995, 2820145, 4210229, 4645143, 5264579\Big\}$$

Theoretical considerations:

- theoretical bound for number of points: $|\{\boldsymbol{k} - \boldsymbol{l} : \boldsymbol{k}, \boldsymbol{l} \in I\}| = 91$
- lattice rules work only from $n = 103$ equidistant points

Numerical experiments: bad frequencies



with weights and equal-weighted

with $\varepsilon = \| \sum_{i=1}^{N+1} \omega_i \boldsymbol{\varphi}(\boldsymbol{x}^i) \cdot \boldsymbol{\varphi}(\boldsymbol{x}^i)^* - \boldsymbol{I} \|_{2\to 2}$

Numerical experiments: “blessing of dimension”

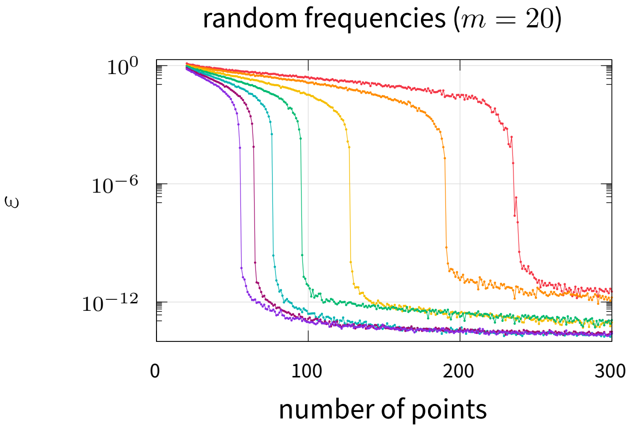
Setting:

- dimension $d = 1, \dots, 7$
- 20 random frequencies in $[-100, 100]^d$

Theoretical considerations:

- theoretical bound for number of points: $|\{\boldsymbol{k} - \boldsymbol{l} : \boldsymbol{k}, \boldsymbol{l} \in I\}| \approx 380$

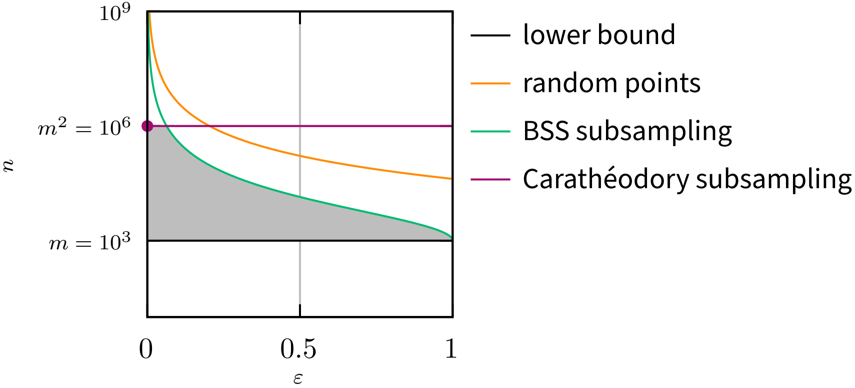
Numerical experiments: “blessing of dimension”



$d = 1$, $d = 2$, $d = 3$, $d = 4$, $d = 5$, $d = 6$, and $d = 7$

with $\varepsilon = \| \sum_{i=1}^{N+1} \omega_i \boldsymbol{\varphi}(\boldsymbol{x}^i) \cdot \boldsymbol{\varphi}(\boldsymbol{x}^i)^* - \boldsymbol{I} \|_{2\to 2}$

Conclusion



number of points for L_2 MZ inequality with constants $1-\varepsilon$ and $1+\varepsilon$ for $\dim(V) = m = 1000$