Minimal Subsampled Rank-1 Lattices for Multivariate Approximation with Optimal Convergence Rate

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Worst-case setting

Given:

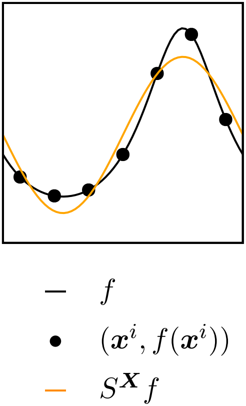
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$
- class of functions F

Goal: find

- points $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\} \subset \mathbb{T}^d$
- efficient algorithm $S^{\boldsymbol{X}}$

with small **worst-case error**

$$\sup_{f \in F} \|f - S^{\boldsymbol{X}} f\|_{L_2}$$



Korobov spaces

Definition

The **weighted Korobov space** is given by

$$H = H_{d,\alpha,\boldsymbol{\gamma}} := \left\{ f \colon \mathbb{T}^d \to \mathbb{C} : \|f\|_H^2 = \sum_{\boldsymbol{h} \in \mathbb{Z}^d} r(\boldsymbol{h}) \, |\hat{f}_{\boldsymbol{h}}|^2 < \infty \right\},$$

with $r(\boldsymbol{h}) = r_{d,\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) := \gamma_{\mathrm{supp}(\boldsymbol{h})}^{-1} \prod_{j \in \mathrm{supp}(\boldsymbol{h})} |h_j|^{2\alpha}$.

special cases include:

- $\gamma_{\mathfrak{u}} = 1$ (unweighted) Sobolev spaces of dominating mixed smoothness
- $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ product weights
- $\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|}$ order-dependent weights
- $\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|} \prod_{j \in \mathfrak{u}} \gamma_j$ POD weights
- $\gamma_{\mathfrak{u}} = \sum_{\boldsymbol{\nu}_{\mathfrak{u}} \in \{1,\dots,\alpha\}^{|\mathfrak{u}|}} \Gamma_{\|\boldsymbol{\nu}_{\mathfrak{u}}\|_1} \prod_{j \in \mathfrak{u}} \gamma_{j,\nu_j}$ SPOD weights

Algorithms

Least squares approximation

input:
- samples $\boldsymbol{X} \subset \mathbb{T}^d$, $f(\boldsymbol{x}^i) \in \mathbb{C}$
- frequencies $\mathcal{B} \subset \mathbb{Z}^d$

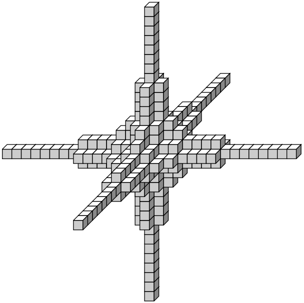
method: $V := \operatorname{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{h}, \cdot\rangle)\}_{\boldsymbol{h}\in\mathcal{B}}$

$$S_{\mathcal{B}}^{\boldsymbol{X}} f := \operatorname*{arg\,min}_{g\in V} \sum_{i=1}^{n} |f(\boldsymbol{x}^i) - g(\boldsymbol{x}^i)|^2$$

$$= \sum_{\boldsymbol{h}\in\mathcal{B}} \hat{g}_{\boldsymbol{h}} \exp(2\pi\mathrm{i}\langle \boldsymbol{h}, \cdot\rangle)$$

computation: $\hat{\boldsymbol{g}} = (\boldsymbol{L}^*\boldsymbol{L})^{-1}\boldsymbol{L}^*\boldsymbol{f}$ with

$$\boldsymbol{L} = [\exp(2\pi\mathrm{i}\langle \boldsymbol{h}, \boldsymbol{x}^i\rangle)]_{i\in\{1,\dots,n\},\boldsymbol{h}\in\mathcal{B}}$$



hyperbolic cross frequency index set

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Kernel method

input: • samples $\boldsymbol{X} \subset \mathbb{T}^d$, $f(\boldsymbol{x}^i) \in \mathbb{C}$
• kernel $K\colon \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{C}$

method:

$$A_{\mathsf{ker}}^{\boldsymbol{X}} f := \sum_{i=1}^{n} a_i K(\cdot, \boldsymbol{x}^i)$$

with $\boldsymbol{a} = \boldsymbol{K}^{-1}\boldsymbol{f}$ and

$$\boldsymbol{K} = [K(\boldsymbol{x}^i, \boldsymbol{x}^j)]_{i,j\in\{1,\dots,n\}}$$

feature: given points $\boldsymbol{X}$, this method has the best worst-case error

Points

Definition: Rank-1 lattice

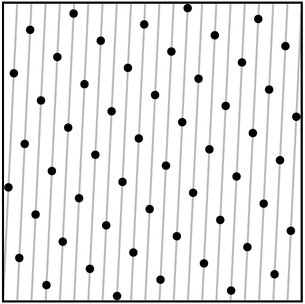
$$\boldsymbol{X} = \left\{ \frac{k}{n} \boldsymbol{z} \bmod 1 \right\}_{k=0}^{n-1} \subset \mathbb{T}^d$$

with **generating vector $\boldsymbol{z}$** and **lattice size n**.

Component-by-component (CBC) algorithm

$z_1 := 1$
`for` $j = 2, \ldots, d$
  `choose` z_j `based on criterion`
`end`

- FFT applicable



$\boldsymbol{z} = (1, 21)^\top, n = 55$

Points

Theorem [Kämmerer, Potts, Volkmer 2015]

Constructing $\boldsymbol{z}$ by CBC fulfilling the **reconstructing property**

$$\langle \boldsymbol{h}, \boldsymbol{z} \rangle \not\equiv_n \langle \boldsymbol{\ell}, \boldsymbol{z} \rangle \quad \forall \boldsymbol{h} \neq \boldsymbol{\ell} \in \mathcal{B}$$

yields for smoothness $\alpha > 1/2$ and $\varepsilon > 0$

$$\sup_{\|f\|_H \leq 1} \|f - S_{\mathcal{B}}^{\boldsymbol{X}} f\|_{L_2} \lesssim n^{-\alpha/2+\varepsilon} .$$

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Theorem [Cools, Kuo, Nuyens, Sloan 2020]

Constructing $\boldsymbol{z}$ by CBC minimizing

$$\mathcal{S}_n(\boldsymbol{z}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^d} \frac{1}{r(\boldsymbol{h})} \sum_{\substack{\boldsymbol{\ell} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\} \\ \langle \boldsymbol{\ell}, \boldsymbol{z} \rangle \equiv_n 0}} \frac{1}{r(\boldsymbol{h} + \boldsymbol{\ell})}$$

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Theorem [Byrenheid, Kämmerer, T. Ullrich, Volkmer 2017]

Any algorithm using lattice points satisfies

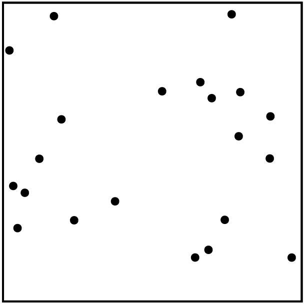
$$\sup_{\|f\|_H \le 1} \|f - S^{\boldsymbol{X}} f\|_{L_2} \gtrsim n^{-\alpha/2}$$

Points

Theorem [Krieg, M. Ullrich 2019]

Let $\boldsymbol{X}$ be n i.i.d. uniform random points.
Then for $\varepsilon > 0$

$$\sup_{\|f\|_H \le 1} \|f - S_{\mathcal{B}}^{\boldsymbol{X}} f\|_{L_2} \lesssim \left(\frac{n}{\log n}\right)^{-\alpha+\varepsilon}.$$



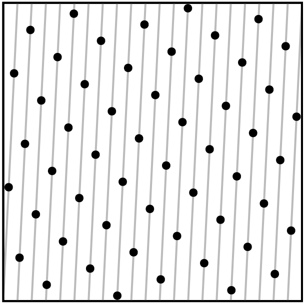
random points

- there is a logarithmic gap, which was closed by [Nagel, Schäfer, T. Ullrich 2022], [Dolbeault, Krieg, M. Ullrich 2022], [B, Schäfer, T. Ullrich 2023]
- no fast algorithm due to lack of structure

Subsampled rank-1 lattices

1

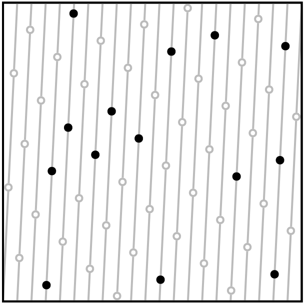
initial rank-1 lattice $\boldsymbol{X}$



$\rightarrow$

2

randomly subsampled $\boldsymbol{X}_J$



Fast algorithm

- initial rank-1 lattice $\boldsymbol{X} = \{\frac{k}{n}\boldsymbol{z} \bmod 1\}_{k=0}^{n-1}$ with $\mathcal{O}(n \log n)$ algorithm
- small point set $\boldsymbol{X}_J = \{\frac{k}{n}\boldsymbol{z} \bmod 1\}_{k\in J} \subset \boldsymbol{X}$

Fast algorithm reusable

We have
- $\boldsymbol{L}_J = \boldsymbol{P}\boldsymbol{L}$ for the least-squares matrix and
- $\boldsymbol{K}_J = \boldsymbol{P}\boldsymbol{K}\boldsymbol{P}^*$ for the kernel matrix with

$$\boldsymbol{P} = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \\ \begin{array}{ccccccc} j_1 & & j_2 & & & j_n & \end{array} \end{array}.$$

- with $|J| \sim \sqrt{n}\log n$ the naive and FFT-approach have the same arithmetic cost

Theoretical guarantees

Theorem [B, Kämmerer, Potts, T. Ullrich 2022]

For unweighted Korobov spaces with $\alpha > 1/2$ there is a subsampled lattice $\boldsymbol{X}_J$ with

$$\sup_{\|f\|_H \le 1} \|f - S_{\mathcal{B}}^{\boldsymbol{X}_J} f\|_{L_2} \lesssim |J|^{-\alpha} (\log |J|)^{(d-1)\alpha + 1/2},$$

and initial lattice size

$$n \sim \left|J\right|^{\frac{2}{1-1/(2\alpha)}}.$$

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and initial lattice size

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Theorem [B, Gilbert, Kuo, Sloan 2025]

For Korobov spaces with $\alpha > 1/2$ there is a subsampled lattice $\boldsymbol{X}_J$ with $\varepsilon > 0$

$$\sup_{\|f\|_H \le 1} \|f - S_{\mathcal{B}}^{\boldsymbol{X}_J} f\|_{L_2} \lesssim \left(\frac{|J|}{\log |J|} \right)^{-\alpha+\varepsilon},$$

and initial lattice size

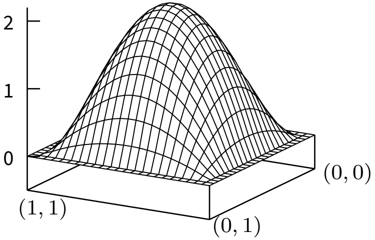
$$n \lesssim \left(\frac{|J|}{\log |J|} \right)^{\frac{2}{\sqrt{1-\varepsilon/\alpha}}}.$$

Note, rate $|J|^{-\alpha+\varepsilon}$ implies $n \gtrsim |J|^{2-\frac{2\varepsilon}{\alpha}}$.

First numerical experiment

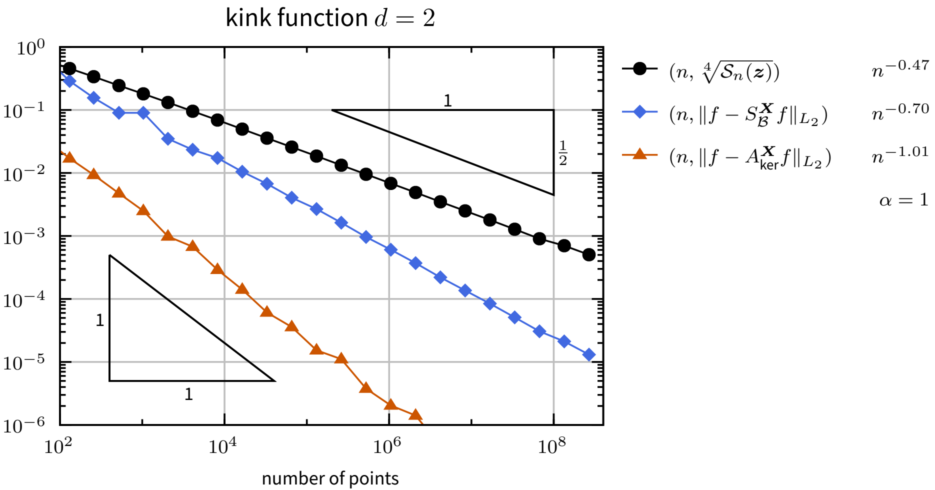
- example function in $d = 2$

$$f(\boldsymbol{x}) = \left(\frac{5^{3/4}15}{4\sqrt{3}}\right)^d \prod_{j=1}^{d} \max\left\{0, \frac{1}{5} - \left(x_j - \frac{1}{2}\right)^2\right\}$$



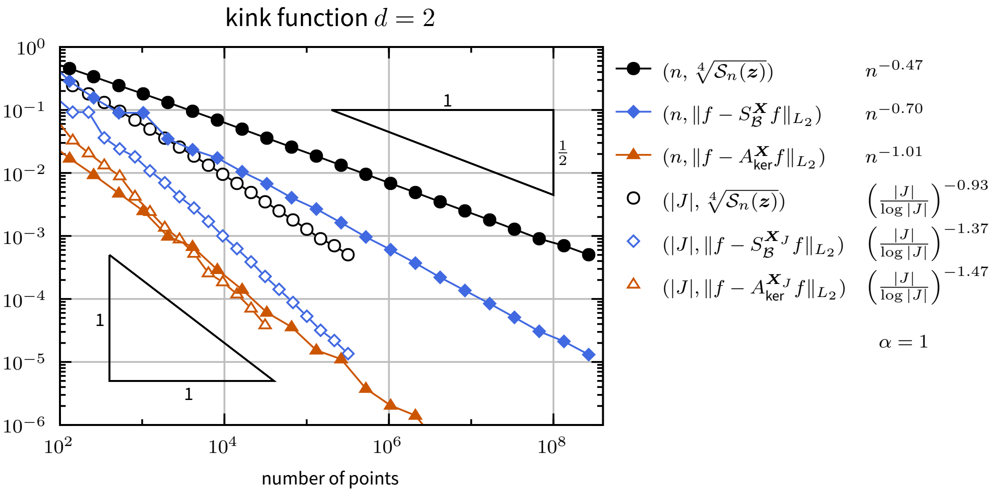
- f has smoothness $3/2 - \varepsilon$
- rank-1 lattice from [Gilbert, Sloan WIP]
- least-squares approximation and kernel method
- full and subsampled rank-1 lattice

First numerical experiment



- [Kaarnioja, Sloan 2025] on doubling the rate

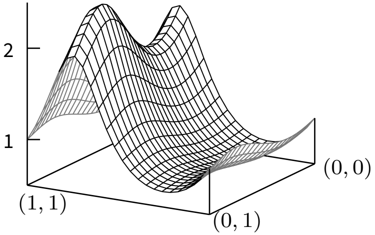
First numerical experiment



Second numerical experiment

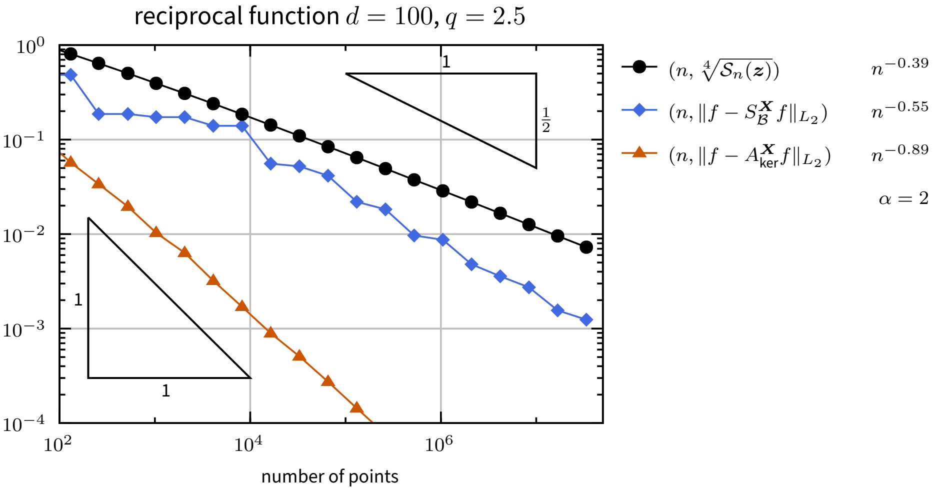
- analytic example function in $d = 100$

$$f(\boldsymbol{x}) = \frac{1}{a(\boldsymbol{x})}\,, \qquad a(\boldsymbol{x}) = 1 + \frac{1}{2}\sum_{j=1}^{d} j^{-2.5}\sin(2\pi x_j)$$



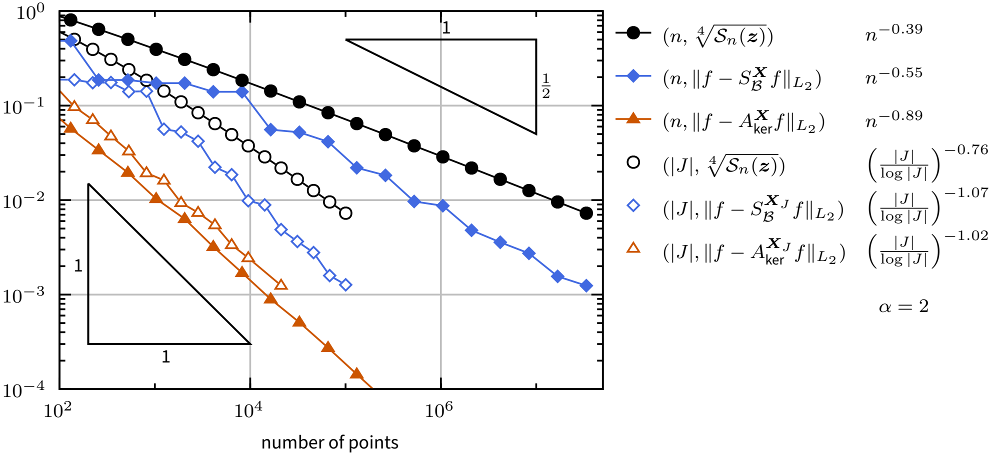
- rank-1 lattice from [Gilbert, Sloan wip]
- least-squares approximation and kernel method
- full and subsampled rank-1 lattice

Second numerical experiment



Second numerical experiment

reciprocal function $d = 100, q = 2.5$



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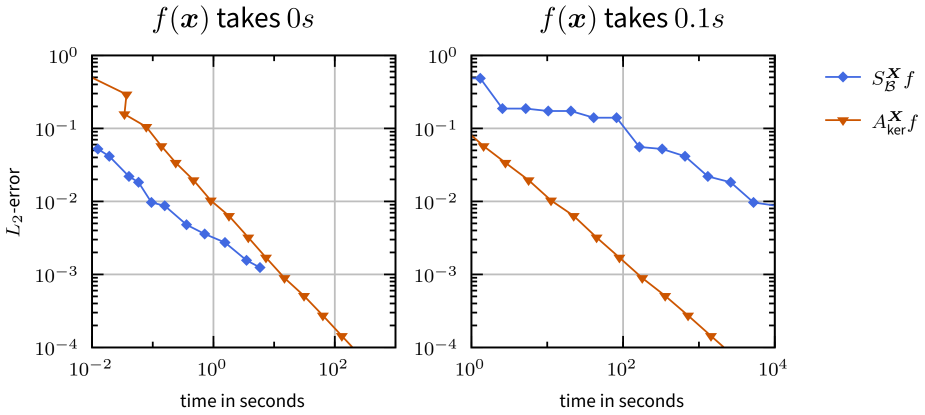

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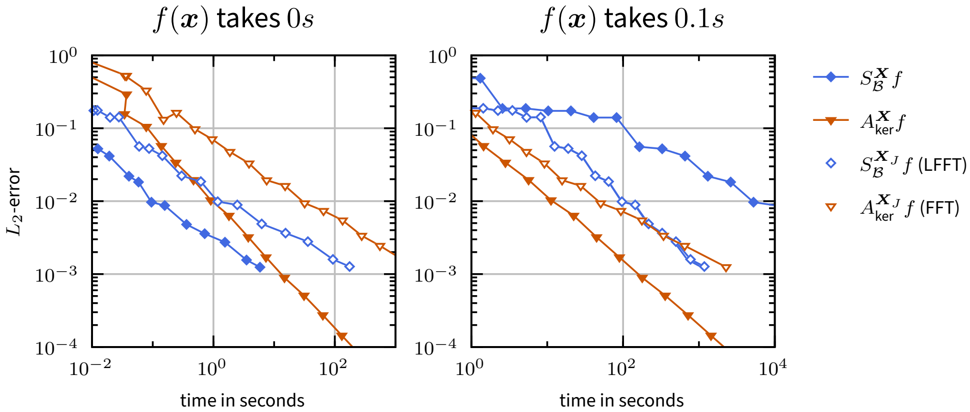
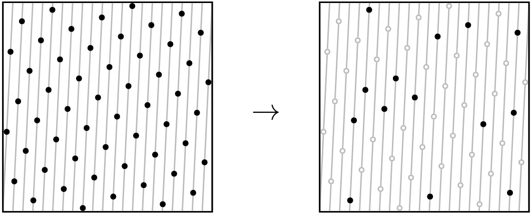


Conclusion



- we propose a **random subsampling** procedure for **rank-1 lattices**
  - achieving near optimal sampling complexity
  - making it possible to utilize FFT algorithms
  - with minimal initial rank-1 lattice size
- [B, Gilbert, Kuo, Sloan] on **arXiv:2506.07729**

Extra: connection of the CBC approaches

Theorem [B, Gilbert, Kuo, Sloan 2025]

Let
- $\boldsymbol{X} = \{\frac{k}{n}\boldsymbol{z} \bmod 1\}_{k=0}^{n-1} \subset \mathbb{T}^d$ and
- $A^{\boldsymbol{X}}$ an approximation algorithm using samples of $\boldsymbol{X}$.

Then $\boldsymbol{X}$ has the **reconstructing property**

$$\langle \boldsymbol{h}, \boldsymbol{z} \rangle \not\equiv_n \langle \boldsymbol{\ell}, \boldsymbol{z} \rangle \quad \forall \boldsymbol{h}, \boldsymbol{\ell} \in \mathcal{B}$$

for

$$\Big\{ \boldsymbol{h} \in \mathbb{Z}^d : r(\boldsymbol{h}) < \Big(\sup_{\|f\|_H \le 1} \|f - A^{\boldsymbol{X}}\|_{L_2} \Big)^{-2} \Big\}.$$