

Sampling recovery in Bochner spaces and applications to parametric PDEs

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Abstract

We prove convergence rates of linear sampling recovery of functions in abstract Bochner spaces satisfying weighted summability of their generalized polynomial chaos expansion coefficients. The underlying algorithm is a function-valued extension of the least squares method widely used and thoroughly studied in scalar-valued function recovery. We apply our theory to two core problems in Computational Uncertainty Quantification. First, we address non-intrusive approximations of solutions to parametric elliptic or parabolic PDEs with log-normal inputs, using a finite set of particular solvers. Second, we consider approximating infinite-dimensional holomorphic functions that arise as solutions to more general parametric PDEs with Gaussian random field inputs. This approach yields substantial improvements in the state of the art for these problems. Importantly, our framework unifies log-normal and affine input models. In the affine case, we obtain convergence rates that improve known results by a logarithmic factor.

Keywords and Phrases: Uncertainty Quantification; Sampling recovery; Bochner spaces; Non-intrusive approximation; Least squares approximation; Parametric PDEs with random inputs; Infinite dimensional holomorphic function; Convergence rate.

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1 Introduction and main results

In Computational Uncertainty Quantification, the problem of efficient approximation for parametric PDEs with random inputs is of great interest and significant progress was achieved in recent

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years. The number of works on this topic is too large to mention all of them. We refer the reader to the works [4, 17, 28, 30, 39] for surveys and bibliography on various aspects on it.

The principal distinction among numerical methods for parametric PDEs is whether they are non-intrusive or intrusive. Non-intrusive methods use an existing solver (exact or approximate) for the PDE, enabling deployment even when the solver is treated as a black box. Intrusive methods, by contrast, incorporate the explicit PDE formulation into the approximation process, demanding complete knowledge of the parametric PDE model. For a detailed comment, see, e.g., [17, Section 1.6].

We are interested in the problem of linear semi-discrete non-intrusive approximation of solutions $u(\mathbf{y})$ to parametric PDEs and the corresponding convergence rate based on a finite number of particular solvers $u(\mathbf{y}_1), \dots, u(\mathbf{y}_n)$ for $\mathbf{y}_1, \dots, \mathbf{y}_n$ from a domain $\mathbb{U}^\infty$, where $\mathbb{U}^\infty$ usually is the infinite-dimensional domain $\mathbb{I}^\infty$ or $\mathbb{R}^\infty$. We do not consider fully discrete (multilevel) approximations which simultaneously treat an approximation discretization on both spatial and parametric domains. The reader can consult [22, 28] for survey and bibliography in this direction. In the present paper we focus on parametric elliptic PDEs with *log-normal* random diffusion coefficients and also consider these equations with affine random diffusion coefficients. Related problems of adaptive nonlinear semi-discrete approximation for parametric PDEs was investigated in [16, 17, 2, 14, 15], and of linear semi-discrete approximation in [5, 6, 21, 22, 23, 29, 35, 37, 38, 28, 41, 42].

Solutions $u(\mathbf{y})$, $\mathbf{y} \in \mathbb{U}^\infty$, to parametric PDEs can be considered as elements of a Bochner space $L_2(\mathbb{U}^\infty, X; \mu)$, where X is a Hilbert space, and μ is a probability measure on $\mathbb{U}^\infty$. The problem, as previously formulated, is equivalent to solving a linear sampling recovery problem in the specified space. We leverage recent breakthrough results for scalar-valued sampling recovery in reproducing kernel Hilbert spaces (RKHS) [11, 27, 32, 33], and transfer them to the Hilbert-valued approximation setting (see surveys and bibliography on related results in [26, 40]). This transferring method is related to, but distinct from, the approach of “lifting” the least squares approximation analysis from scalar-valued to Hilbert-valued settings studied in [2, 3, 5, 14, 20], where direct least squares algorithms are employed for sampling recovery of Hilbert-valued functions. In contrast, the present work reduces the Hilbert-valued sampling recovery problem to a scalar-valued one, which is a crucial difference from the lifting framework described in the cited papers.

Before presenting the main result on sampling recovery in Bochner spaces, we need to comment on the setup. Let (U, Σ, μ) be a probability measure space with U being a separable topological space and let X be a complex separable Hilbert space. Denote by $L_2(U, X; \mu) = L_2(U, \mathbb{C}; \mu) \otimes X$ the Bochner space of strongly μ -measurable mappings v from U to X , equipped with the norm

$$\|v\|_{L_2(U, X; \mu)} := \left(\int_U \|v(\mathbf{y})\|_X^2 d\mu(\mathbf{y}) \right)^{1/2}.$$

Notice that because U and X are separable, so is $L_2(U, X; \mu)$. We fix $(\varphi_s)_{s \in \mathbb{N}}$, an orthonormal basis of $L_2(U, \mathbb{C}; \mu)$. Then a function $v \in L_2(U, X; \mu)$ can be represented by the expansion

$$v(\mathbf{y}) = \sum_{s \in \mathbb{N}} v_s \varphi_s(\mathbf{y}) \quad \text{with} \quad v_s := \int_U v(\mathbf{y}) \overline{\varphi_s(\mathbf{y})} d\mu(\mathbf{y}) \in X \quad (1.1)$$

with convergence in $L_2(U, X; \mu)$. Moreover, for every $v \in L_2(U, X; \mu)$ represented by the series (1.1), Parseval’s identity holds

$$\|v\|_{L_2(U, X; \mu)}^2 = \sum_{s \in \mathbb{N}} \|v_s\|_X^2.$$

Throughout the present paper, we fix $(\sigma_s)_{s \in \mathbb{N}}$, a non-decreasing sequence of positive numbers such that $\boldsymbol{\sigma}^{-1} := (\sigma_s^{-1})_{s \in \mathbb{N}} \in \ell_2(\mathbb{N})$. For given U and μ , denote by $H_{X,\boldsymbol{\sigma}}$ the linear subspace in $L_2(U, X; \mu)$ of all v such that the norm

$$\|v\|_{H_{X,\boldsymbol{\sigma}}} := \left(\sum_{s \in \mathbb{N}} (\sigma_s \|v_s\|_X)^2 \right)^{1/2} < \infty.$$

In particular, the space $H_{\mathbb{C},\boldsymbol{\sigma}}$ is the linear subspace in $L_2(U, \mathbb{C}; \mu)$ equipped with its own inner product

$$\langle f, g \rangle_{H_{\mathbb{C},\boldsymbol{\sigma}}} := \sum_{s \in \mathbb{N}} \sigma_s^2 \langle f, \varphi_s \rangle_{L_2(U, \mathbb{C}; \mu)} \overline{\langle g, \varphi_s \rangle_{L_2(U, \mathbb{C}; \mu)}}$$

and forms a reproducing kernel Hilbert space with the reproducing kernel

$$K(\cdot, \mathbf{y}) := \sum_{s \in \mathbb{N}} \sigma_s^{-2} \varphi_s(\cdot) \overline{\varphi_s(\mathbf{y})}$$

with eigenfunctions $(\varphi_s)_{s \in \mathbb{N}}$ and eigenvalues $(\sigma_s^{-1})_{s \in \mathbb{N}}$. Moreover, $K(\mathbf{x}, \mathbf{y})$ satisfies the finite trace assumption

$$\int_U K(\mathbf{x}, \mathbf{x}) d\mu(\mathbf{x}) < \infty,$$

which is not only natural in sampling recovery but also crucial to the techniques we employ. We refer readers, e.g., to [12, 36] for necessary background on RKHSs.

We study the approximate recovery of functions in the space $H_{X,\boldsymbol{\sigma}}$ from a finite set of their samples. To ensure a coherent formulation of the problem, we begin with a preliminary observation.

From the separability of X it follows that there exists a set $U_0 \subset U$ satisfying $\mu(U \setminus U_0) = 0$ such that

$$K(\mathbf{x}, \mathbf{y}) := \sum_{s \in \mathbb{N}} \sigma_s^{-2} \varphi_s(\mathbf{x}) \overline{\varphi_s(\mathbf{y})}, \quad \forall \mathbf{x}, \mathbf{y} \in U_0,$$

and

$$f(\mathbf{y}) = \sum_{s \in \mathbb{N}} \sigma_s^{-2} \langle f, \varphi_s \rangle_{H_{X,\boldsymbol{\sigma}}} \varphi_s(\mathbf{y}), \quad \forall f \in L_2(U, \mathbb{C}; \mu), \quad \forall \mathbf{y} \in U_0.$$

This means that the pointwise evaluations $f(\mathbf{y})$ are well-defined for every $\mathbf{y}$ belonging the full measure subset U_0 of U .

Let $(\varphi_k)_{k \in \mathbb{N}}$ be an orthonormal basis of X . If $v \in H_{X,\boldsymbol{\sigma}}$, then $\langle v, \psi_k \rangle_X \in H_{\mathbb{C},\boldsymbol{\sigma}}$ for every k . Consequently, the pointwise evaluations $\langle v(\mathbf{y}), \psi_k \rangle_X$ are well-defined for every $\mathbf{y} \in U_0$. This implies that the pointwise evaluations $v(\mathbf{y})$ are also well-defined for every $\mathbf{y} \in U_0$.

Throughout this paper, in the context of sampling recovery, the inclusion $v \in H_{X,\boldsymbol{\sigma}}$ means that v is a representative of an element from $L_2(U, X; \mu)$ that the pointwise evaluations $v(\mathbf{y})$ for every $\mathbf{y} \in U_0$ are well-defined with U_0 fixed.

Let us formulate the problem of sampling recovery for $v \in H_{X,\boldsymbol{\sigma}}$ as follows: Given sample points $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and functions $h_1, \dots, h_n \in L_2(U, \mathbb{C}; \mu)$, we consider the approximate recovery of v from its values $v(\mathbf{y}_1), \dots, v(\mathbf{y}_n)$ by the linear sampling algorithm S_n^X on U defined as

$$(S_n^X v)(\mathbf{y}) := \sum_{i=1}^n v(\mathbf{y}_i) h_i(\mathbf{y}). \tag{1.2}$$

For convenience, we assume that some of the sample points $\mathbf{y}_i$ may coincide.

Denote

$$B_{X,\sigma} := \left\{ v \in H_{X,\sigma} : \|v\|_{H_{X,\sigma}} \leq 1 \right\},$$

the unit ball in $H_{X,\sigma}$.

Let $\mathcal{S}_n^X$ be the family of all linear sampling algorithms S_k^X in $L_2(U, X; \mu)$ of the form (1.2) with $k \leq n$. To study the optimality of linear sampling algorithms from $\mathcal{S}_n^X$ for the set $B_{X,\sigma}$ and their convergence rates we use the (linear) sampling n -width

$$\varrho_n(B_{X,\sigma}, L_2(U, X; \mu)) := \inf_{S_n^X \in \mathcal{S}_n^X} \sup_{v \in B_{X,\sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)}.$$

Let us describe the main contribution of the present paper.

We establish convergence rates for an extension of the least squares method with varying sampling strategies with the following basic result. For the sampling algorithm S_n^X in $L_2(U, X; \mu)$ defined by (1.2), we have

$$\sup_{v \in B_{X,\sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} = \sup_{f \in B_{\mathbb{C},\sigma}} \|f - S_n^{\mathbb{C}} f\|_{L_2(U, \mathbb{C}; \mu)},$$

and, hence,

$$\varrho_n(B_{X,\sigma}, L_2(U, X; \mu)) = \varrho_n(B_{\mathbb{C},\sigma}, L_2(U, \mathbb{C}; \mu)). \quad (1.3)$$

This relation makes available bounds on the sampling widths in the classical Lebesgue space $L_2(U, \mathbb{C}; \mu)$ applicable to a general Bochner space $L_2(U, X; \mu)$.

From the equality (1.3) and an inequality between the sampling widths and Kolmogorov widths proven in [27, Theorem 1] we derive the optimal convergence rate of $\varrho_n(B_{X,\sigma}, L_2(U, X; \mu))$ for $0 < q < 2$ in the sense that the relations

$$n^{-1/q} \leq \sup_{\sigma: \|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1} \varrho_n(B_{X,\sigma}, L_2(U, X; \mu)) \asymp n^{-1/q} \quad (1.4)$$

hold. In particular, it holds

$$\sup_{\sigma: \|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1} \varrho_n(B_{X,\sigma}, L_2(\mathbb{U}^\infty, X; \boldsymbol{\mu})) \asymp n^{-1/q} \quad (1.5)$$

for the Bochner space $L_2(\mathbb{U}^\infty, X; \boldsymbol{\mu})$ with $\boldsymbol{\mu}$, the infinite tensor-product standard Gaussian measure on $\mathbb{U}^\infty = \mathbb{R}^\infty$, or the infinite tensor-product Jacobi probability measure on $\mathbb{U}^\infty = \mathbb{I}^\infty$. It is worth mentioning that the underlying sampling algorithm performing the convergence rate in (1.4) and (1.5) is an extension to Bochner spaces of a classical least squares approximation with a non-constructive subsampling. Regarding the constructiveness of linear sampling algorithms, similar extensions of a classical least squares approximation and of a least squares approximation with a special constructive subsampling, give the error bounds $n^{-1/q}(\log n)^{1/q}$ and $n^{-1/q}(\log n)^{1/2}$, respectively. Thanks to this constructive subsampling, the cost of computation is significantly reduced for a sufficiently large number of sample points (for details, see [11]).

We apply these approximation results in general Bochner spaces to the formulated linear approximation for parametric PDEs with affine or log-normal inputs as well as for infinite-dimensional

holomorphic functions, significantly improving existing convergence rates. Moreover, differing from the previous papers mentioned above which usually considered the affine and log-normal cases of random inputs separately, with this approach we treat both cases together by employing a *unified method*.

The specific setting for the linear approximation problem for a wide class of parametric PDEs with random inputs as well as for infinite-dimensional holomorphic functions goes as follows. Under a certain condition the weak parametric solution $u(\mathbf{y})$ to a parametric elliptic PDE equation with log-normal ($\mathbb{U}^\infty = \mathbb{R}^\infty$) or affine ($\mathbb{U}^\infty = \mathbb{I}^\infty$) random inputs, satisfies a weighted ℓ_2 -summability of the energy norms of the Hermite or Jacobi GPC expansion coefficients, respectively, in terms of the inclusion $u(\mathbf{y}) \in MB_{V,\sigma}$ with $\|\sigma^{-1}\|_{\ell_q(\mathbb{F})} \leq N$ for some $0 < q < 2$, $M, N > 0$ and positive sequences σ , where $V := H_0^1(D)$ is the energy space and D is the spatial domain (see Lemmata 3.1 and 3.2 below). This allows us to apply all the above results for abstract Bochner spaces to parametric elliptic or parabolic PDEs. The most significant application is that, from (1.5), there exists a linear sampling algorithm S_n^V in $L_2(\mathbb{U}^\infty, V; \mu)$ of the form (1.2) for $X = V$ such that

$$\|u - S_n^V u\|_{L_2(\mathbb{U}^\infty, V; \mu)} \leq CMNn^{-1/q},$$

where C is a positive constant independent of M, N, n and u . For log-normal random inputs ($\mathbb{U}^\infty = \mathbb{R}^\infty$), the convergence rate of linear non-instructive approximation of the parametric solution $u(\mathbf{y})$ obtained via the sampling algorithm S_n^V is $n^{-1/q}$. This is an important improvement compared to all previous works [6, 21, 22, 23, 28, 29, 35, 37, 38], etc., which are off by a factor $n^{1/2}$ from this convergence rate. In particular, it is significantly better than the rates $n^{-\frac{1}{2}(1/q-1/2)}$ and $n^{-(1/q-1/2)}$ which have been recently obtained in [29, Theorem 3.18] and [22, Corollary 5.3], respectively. The same notable improvement of convergence rate holds true for linear polynomial interpolation approximation of relevant infinite-dimensional holomorphic functions on $\mathbb{R}^\infty$ (cf. [28]) and of parametric parabolic PDEs. In the case of affine random inputs ($\mathbb{U}^\infty = \mathbb{I}^\infty$), the convergence rate $n^{-1/q}$ is better than the best-known convergence rate $(n/\log n)^{-1/q}$ of (linear and non-linear) non-instructive approximation (cf. [5, 14, 21, 41]). Notice also that the convergence rate $n^{-1/q}$ coincides with the optimal convergence rate of (linear and non-linear) intrusive spectral and Galerkin approximation of solutions to parametric PDEs with random inputs (cf. [7, 8, 9, 22]). We believe that, using the techniques developed in this paper, similar improvements could also be achieved for the relevant linear approximation of infinite-dimensional holomorphic functions on $\mathbb{I}^\infty$. However, this is beyond the scope of current consideration. Last but not least, it is worth emphasizing that the proof methods for the main results are simple yet entirely novel. The approaches rely on the connection between sampling recovery in abstract RKHSs and associated approximations of solutions to parametric PDEs. The proof methods are then based on the weighted summability properties of the generalized polynomial chaos expansion coefficients. Moreover, these methods can be extended and generalized to other problems in Uncertainty Quantification for parametric PDEs with random inputs such as fully discrete non-instructive approximation, among others. The last problem has been recently considered in [25].

Note that in all convergence rates reported in this work, the sequence σ is assumed known. It determines the selected basis $(\varphi_s)_{s \in \mathbb{N}}$ and, consequently, the least squares sampling algorithm. The case in which σ is unknown has been treated in [5], wherein the problem is approached via compressed-sensing techniques rather than the lifting method.

The rest of the paper is organized as follows. In Section 2, we investigate sampling recovery in

abstract Bochner spaces, in particular, with infinite-dimensional measure. Here, we present some least squares methods and their extensions to Bochner spaces. In Section 3 and 4, we apply the results of Section 2 to linear approximation for parametric elliptic PDE equation with affine or log-normal random inputs, and for infinite-dimensional holomorphic functions on $\mathbb{R}^\infty$, respectively. In Section 5, we discuss constructiveness and alternative least squares methods which can be applied to non-intrusive approximations for parametric PDEs and infinite-dimensional holomorphic functions.

Notations: As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers, and $\mathbb{N}_0 := \{s \in \mathbb{Z} : s \geq 0\}$. We denote $\mathbb{R}^\infty$ and $\mathbb{I}^\infty := [-1, 1]^\infty$ the sets of all sequences $\mathbf{y} = (y_j)_{j \in \mathbb{N}}$ with $y_j \in \mathbb{R}$ and $y_j \in [-1, 1]$, respectively, and $\mathbb{C}^\infty$ the set of all sequences $\mathbf{z} = (z_j)_{j \in \mathbb{N}}$ with $z_j \in \mathbb{C}$. Denote by $\mathbb{F}$ the set of all sequences of non-negative integers $\mathbf{s} = (s_j)_{j \in \mathbb{N}}$ such that their support $\text{supp}(\mathbf{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. If $\mathbf{a} = (a_j)_{j \in \mathcal{J}}$ is a set of positive numbers with any index set $\mathcal{J}$, then we use the notation $\mathbf{a}^{-1} := (a_j^{-1})_{j \in \mathcal{J}}$. For $\mathbf{s}, \mathbf{s}' \in \mathbb{F}$ and $\mathbf{y} \in \mathbb{R}^\infty$, denote: $\mathbf{s}! := \prod_{j \in \mathbb{N}} s_j!$; $\mathbf{y}^{\mathbf{s}} := \prod_{j \in \mathbb{N}} y_j^{s_j}$; $\binom{\mathbf{s}}{\mathbf{s}'} := \prod_{j \in \mathbb{N}} \binom{s_j}{s'_j}$. We use letter C to denote general positive constants which may take different values, and $C_{a,b,\dots}$ a positive constant depending on $a, b, \dots$. For the quantities $A_n(f, \mathbf{k})$ and $B_n(f, \mathbf{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\mathbf{k} \in \mathbb{Z}^d$, we write $A_n(f, \mathbf{k}) \lesssim B_n(f, \mathbf{k})$, $f \in W$, $\mathbf{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ such that $A_n(f, \mathbf{k}) \leq C B_n(f, \mathbf{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\mathbf{k} \in \mathbb{Z}^d$ (the notation $A_n(f, \mathbf{k}) \gtrsim B_n(f, \mathbf{k})$ has the obvious opposite meaning), and $A_n(f, \mathbf{k}) \asymp B_n(f, \mathbf{k})$ if $A_n(f, \mathbf{k}) \lesssim B_n(f, \mathbf{k})$ and $B_n(f, \mathbf{k}) \lesssim A_n(f, \mathbf{k})$. Denote by $|G|$ the cardinality of the set G .

2 Sampling recovery in Bochner spaces

In this section, we show that the problem of linear sampling recovery of functions in the space $H_{X,\sigma}$ for a general separable Hilbert space X can be reduced to the particular case of the RKHS $H_{\mathbb{C},\sigma}$. This allows, in particular, to extend linear least squares sampling algorithms in $H_{\mathbb{C},\sigma}$ to $H_{X,\sigma}$ while preserving the accuracy of approximation. Hence, we are able to derive convergence rates of various extended linear least squares sampling algorithms for functions in $B_{X,\sigma}$ based on some recent results on inequalities between sampling widths and Kolmogorov widths of the unit ball $B_{\mathbb{C},\sigma}$ which are realized by linear least squares sampling algorithms.

2.1 Extension of least squares approximation to Bochner spaces

Recall that, $(\varphi_s)_{s \in \mathbb{N}}$ is a fixed orthonormal basis of $L_2(U, \mathbb{C}; \mu)$ and for a function $v \in L_2(U, X; \mu)$ the expansion (1.1) holds, with coefficients $(v_s)_{s \in \mathbb{N}}$ defined as in (1.1). We will need the following auxiliary result. Let A^X be a general linear operator in $L_2(U, X; \mu)$ defined for $v \in L_2(U, X; \mu)$ by

$$A^X v := \sum_{k \in \mathbb{N}} \left(\sum_{s \in \mathbb{N}} a_{k,s} v_s \right) \varphi_k, \quad (2.1)$$

where $(a_{k,s})_{(k,s) \in \mathbb{N}^2}$ is an infinite-dimensional matrix.

Lemma 2.1 *Let A^X be linear and bounded. There holds the equality*

$$\|A^X\|_{H_{X,\sigma} \rightarrow L_2(U, X; \mu)} = \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}.$$

Proof. For a function $f \in H_{\mathbb{C},\sigma}$, we have the series expansion

$$f = \sum_{s \in \mathbb{N}} f_s \varphi_s \quad \text{with} \quad (\sigma_s |f_s|)_{s \in \mathbb{N}} \in \ell_2(\mathbb{N}),$$

and by the operator norm in the classical Lebesgue space

$$\|A^{\mathbb{C}} f\|_{L_2(U, \mathbb{C}; \mu)}^2 \leq \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}^2 \|f\|_{H_{\mathbb{C},\sigma}}^2.$$

Using the series expansion of f and the definition of $A^{\mathbb{C}}$ from (2.1), the last inequality is equivalent to

$$\sum_{k \in \mathbb{N}} \left| \sum_{s \in \mathbb{N}} a_{k,s} f_s \right|^2 \leq \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}^2 \sum_{s \in \mathbb{N}} \sigma_s^2 |f_s|^2. \quad (2.2)$$

Because this is independent of the choice of f , it holds for all sequences $(\sigma_s |f_s|)_{s \in \mathbb{N}} \in \ell_2(\mathbb{N})$.

For the Hilbert-valued case $v \in H_{X,\sigma}$, we have

$$v = \sum_{s \in \mathbb{N}} v_s \varphi_s \quad \text{with} \quad (\sigma_s \|v_s\|_X)_{s \in \mathbb{N}} \in \ell_2(\mathbb{N}),$$

and

$$\|A^X v\|_{L_2(U, X; \mu)}^2 = \sum_{k \in \mathbb{N}} \left\| \sum_{s \in \mathbb{N}} a_{k,s} v_s \right\|_X^2.$$

Let $(\psi_j)_{j \in \mathbb{N}}$ be an orthonormal basis of X and

$$v_s = \sum_{j \in \mathbb{N}} (v_s)_j \psi_j.$$

Then,

$$A^X v = \sum_{k \in \mathbb{N}} \left(\sum_{s \in \mathbb{N}} \sum_{j \in \mathbb{N}} a_{k,s} (v_s)_j \psi_j \right) \varphi_k.$$

Since $(\varphi_k \psi_j)_{k,j \in \mathbb{N}}$ is an orthonormal basis of $L_2(U, X; \mu)$, by applying (2.2) to $f_s = (v_s)_j$, we obtain

$$\begin{aligned} \|A^X v\|_{L_2(U, X; \mu)}^2 &= \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \left| \sum_{s \in \mathbb{N}} a_{k,s} (v_s)_j \right|^2 \\ &\leq \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}^2 \sum_{j \in \mathbb{N}} \sum_{s \in \mathbb{N}} \sigma_s^2 |(v_s)_j|^2 \\ &= \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}^2 \sum_{s \in \mathbb{N}} \sigma_s^2 \|v_s\|_X^2 = \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}^2 \|v\|_{H_{X,\sigma}}^2. \end{aligned}$$

This proves the inequality

$$\|A^X\|_{H_{X,\sigma} \rightarrow L_2(U, X; \mu)} \leq \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}.$$

In order to prove the inverse inequality, let $(f^{(n)})_{n \in \mathbb{N}} \subset H_{\mathbb{C}, \sigma}$ be a sequence such that $\|f^{(n)}\|_{H_{\mathbb{C}, \sigma}} = 1$ and

$$\lim_{n \rightarrow \infty} \|A^{\mathbb{C}} f^{(n)}\|_{L_2(U, \mathbb{C}; \mu)} = \|A^{\mathbb{C}}\|_{H_{\mathbb{C}, \sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}.$$

Define $v^{(n)} := f^{(n)} \psi_1$. Then $\|v^{(n)}\|_{H_{X, \sigma}} = 1$ and

$$\begin{aligned} \|A^X v^{(n)}\|_{L_2(U, X; \mu)}^2 &= \sum_{k \in \mathbb{N}} \left\| \sum_{s \in \mathbb{N}} a_{k,s} \langle f^{(n)}, \varphi_s \rangle_{L_2(U, \mathbb{C}; \mu)} \eta_1 \right\|_X^2 = \sum_{k \in \mathbb{N}} \left| \sum_{s \in \mathbb{N}} a_{k,s} \langle f^{(n)}, \varphi_s \rangle_{L_2(U, \mathbb{C}; \mu)} \right|^2 \\ &= \|A^{\mathbb{C}} f^{(n)}\|_{L_2(U, \mathbb{C}; \mu)}^2 \rightarrow \|A^{\mathbb{C}}\|_{H_{\mathbb{C}, \sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}^2 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This proves the inequality

$$\|A^X\|_{H_{X, \sigma} \rightarrow L_2(U, X; \mu)} \geq \|A^{\mathbb{C}}\|_{H_{\mathbb{C}, \sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}.$$

□

The following theorem plays a key role in transferring current results on linear sampling recovery of functions in RKHSs [11, 27, 32, 33], into semi-discrete non-intrusive approximation for parametric PDEs with random inputs, based on finite number of particular solvers.

Recall that $U_0 \subset U$ is a fixed set of full measure such that if $v \in H_{X, \sigma}$, the pointwise evaluations $v(\mathbf{y})$ are well-defined for every $\mathbf{y} \in U_0$.

Theorem 2.1 *Given arbitrary sample points $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and functions $h_1, \dots, h_n \in L_2(U, \mathbb{C}; \mu)$, for the sampling algorithm S_n^X in $L_2(U, X; \mu)$ defined by (1.2), we have that*

$$\sup_{v \in B_{X, \sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} = \sup_{f \in B_{\mathbb{C}, \sigma}} \|f - S_n^{\mathbb{C}} f\|_{L_2(U, \mathbb{C}; \mu)}, \quad (2.3)$$

moreover,

$$\varrho_n(B_{X, \sigma}, L_2(U, X; \mu)) = \varrho_n(B_{\mathbb{C}, \sigma}, L_2(U, \mathbb{C}; \mu)). \quad (2.4)$$

Proof. Denote by I^X the identity operator in $L_2(U, X; \mu)$. Let S_n^X be an arbitrary sampling operator in $L_2(U, X; \mu)$ given for $v \in L_2(U, X; \mu)$ by

$$S_n^X v(\mathbf{y}) := \sum_{i=1}^n v(\mathbf{y}_i) h_i(\mathbf{y}).$$

Applying Lemma 2.1 with $A^X := I^X - S_n^X$, we get

$$\|I^X - S_n^X\|_{H_{X, \sigma} \rightarrow L_2(U, X; \mu)} = \|I^{\mathbb{C}} - S_n^{\mathbb{C}}\|_{H_{\mathbb{C}, \sigma} \rightarrow L_2(U, \mathbb{C}; \mu)}.$$

Consequently, we obtain (2.3).

Since the correspondence between S_n^X and $S_n^{\mathbb{C}}$ is one-to-one, we use (2.3) to show

$$\inf_{S_n^X \in \mathcal{S}_n^X} \sup_{v \in B_{X, \sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} = \inf_{S_n^{\mathbb{C}} \in \mathcal{S}_n^{\mathbb{C}}} \sup_{f \in B_{\mathbb{C}, \sigma}} \|f - S_n^{\mathbb{C}} f\|_{L_2(U, \mathbb{C}; \mu)},$$

which proves (2.4). $\square$

Let us construct an extension of a least squares approximation in the space $L_2(U, \mathbb{C}; \mu)$ to a space $L_2(U, X; \mu)$. For $n, m \in \mathbb{N}$ with $n \geq m$, let $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ be points, $\omega_1, \dots, \omega_n \geq 0$ be weights, and $V_m = \text{span}\{\varphi_j\}_{j=1}^m$ the subspace spanned by the functions φ_j , $j = 1, \dots, m$. The weighted least squares approximation

$$S_n^{\mathbb{C}} f = S_n^{\mathbb{C}}(\mathbf{y}_1, \dots, \mathbf{y}_n, \omega_1, \dots, \omega_n, V_m) f$$

of a function $f: U \rightarrow \mathbb{C}$ is given by

$$S_n^{\mathbb{C}} f = \arg \min_{g \in V_m} \sum_{i=1}^n \omega_i |f(\mathbf{y}_i) - g(\mathbf{y}_i)|^2. \quad (2.5)$$

The least squares approximation can be computed using the Moore-Penrose inverse, and it is the approximation of smallest error for over-determined systems where no exact solution can be expected. In particular, for $\mathbf{L} = [\varphi_s(\mathbf{y}_i)]_{i=1, \dots, n; s=1, \dots, m}$ and $\mathbf{W} = \text{diag}(\omega_1, \dots, \omega_n)$ we have

$$S_n^{\mathbb{C}} f = \sum_{s=1}^m \hat{g}_s \varphi_s \quad \text{with} \quad (\hat{g}_1, \dots, \hat{g}_m)^{\top} = (\mathbf{W}^{1/2} \mathbf{L})^+ \mathbf{W}^{1/2} (f(\mathbf{y}_1), \dots, f(\mathbf{y}_n))^{\top}, \quad (2.6)$$

where $(\mathbf{W}^{1/2} \mathbf{L})^+$ denotes the pseudo-inverse of $\mathbf{W}^{1/2} \mathbf{L}$. In the presented theorems the setting is such that the matrix $\mathbf{W}^{1/2} \mathbf{L}$ has full rank and we have $(\mathbf{W}^{1/2} \mathbf{L})^+ = (\mathbf{L}^* \mathbf{W} \mathbf{L})^{-1} \mathbf{L}^* \mathbf{W}^{1/2}$. For every $n \in \mathbb{N}$, let

$$S_n^{\mathbb{C}} f := \sum_{i=1}^n f(\mathbf{y}_i) h_i, \quad (2.7)$$

be the least squares sampling algorithm constructed as in (2.5)–(2.6) for these sample points and weights, where $h_1, \dots, h_n \in L_2(U, \mathbb{C}; \mu)$. Hence we immediately obtain the extension of this least squares algorithm to the Bochner space $L_2(U, X; \mu)$ by replacing $f \in L_2(U, \mathbb{C}; \mu)$ with $v \in L_2(U, X; \mu)$:

$$S_n^X v := S_n^X(\mathbf{y}_1, \dots, \mathbf{y}_n, \omega_1, \dots, \omega_n, V_m) v := \sum_{i=1}^n v(\mathbf{y}_i) h_i. \quad (2.8)$$

As the least squares approximation is a linear operator, worst-case error bounds carry over from the usual Lebesgue space $L_2(U, \mathbb{C}; \mu)$ to the Bochner space $L_2(U, X; \mu)$.

Let $n \in \mathbb{N}$ and E be a normed space and F a central symmetric compact set in E . Then the Kolmogorov n -width of F is defined by

$$d_n(F, E) := \inf_{L_n} \sup_{f \in F} \inf_{g \in L_n} \|f - g\|_E,$$

where the left-most infimum is taken over all subspaces L_n of dimension at most n in E .

We make use of the abbreviation $d_n := d_n(B_{\mathbb{C}, \sigma}, L_2(U, \mathbb{C}; \mu))$. In our setting, we know that $d_n = \sigma_{n+1}^{-1}$.

From Theorem 2.1 we can derive the following lemma, which extends to Bochner spaces the recent important result of [27, Theorem 1] on an inequality between sampling widths and Kolmogorov widths in RKHSs.

Lemma 2.2 For any $n \in \mathbb{N}$, there exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and weights $\omega_1, \dots, \omega_n$ such that

$$\sup_{v \in B_{X, \sigma}} \left\| v - \tilde{S}_n^X v \right\|_{L_2(U, X; \mu)}^2 \leq 433 \max \left\{ d_{\lfloor n/43200 \rfloor}^2, \frac{43200}{n} \sum_{s \geq \lfloor n/43200 \rfloor} d_s^2 \right\},$$

where

$$\tilde{S}_n^X := S_n^X(\mathbf{y}_1, \dots, \mathbf{y}_n, \omega_1, \dots, \omega_n, V_m) \quad \text{and} \quad n \geq m \geq \frac{n}{43200}. \quad (2.9)$$

Proof. For the particular case when $X = \mathbb{C}$, this theorem is implied immediately from [27, Theorem 23]. Hence, by using Theorem 2.1 we prove the lemma. $\square$

2.2 Convergence rates

Lemma 2.3 Let $\|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1$ for some $0 < q \leq 2$. Then we have that

$$d_n(B_{\mathbb{C}, \sigma}, L_2(U, \mathbb{C}; \mu)) \leq (n+1)^{-1/q} \quad \forall n \in \mathbb{N}. \quad (2.10)$$

Proof. For $\xi > 0$, we introduce the set

$$\Lambda(\xi) := \{s \in \mathbb{N} : \sigma_s^q \leq \xi\}.$$

For a function $f \in B_{\mathbb{C}, \sigma}$ represented by the series (1.1), we define the truncation

$$S_{\Lambda(\xi)} f := \sum_{s \in \Lambda(\xi)} f_s \varphi_s. \quad (2.11)$$

Applying the Parseval's identity, noting (2.11), we obtain

$$\begin{aligned} \|f - S_{\Lambda(\xi)} f\|_{L_2(U, \mathbb{C}; \mu)}^2 &= \sum_{\sigma: \sigma_s > \xi^{1/q}} |f_s|^2 = \sum_{\sigma: \sigma_s > \xi^{1/q}} (\sigma_s |f_s|)^2 \sigma_s^{-2} \\ &\leq \xi^{-2/q} \sum_{s \in \mathbb{N}} (\sigma_s |f_s|)^2 \leq \xi^{-2/q}. \end{aligned}$$

The function $S_{\Lambda(\xi)} f$ belongs to the linear subspace $L(\xi) := \text{span}\{\varphi_s : s \in \Lambda(\xi)\}$ in $L_2(U, \mathbb{C}; \mu)$ of dimension $|\Lambda(\xi)|$. We have

$$|\Lambda(\xi)| \leq \sum_{\sigma: \sigma_s \leq \xi^{1/q}} 1 \leq \xi \sum_{s \in \mathbb{N}} \sigma_s^{-q} \leq \xi$$

as $\|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1$. For a given $n \in \mathbb{N}$,

$$d_n(B_{\mathbb{C}, \sigma}, L_2(U, \mathbb{C}; \mu)) \leq \|f - S_{\Lambda(\xi)} f\|_{L_2(U, \mathbb{C}; \mu)} \leq \xi^{-1/q}$$

for arbitrary $\xi < n+1$ satisfying the inequality $|\Lambda(\xi)| \leq n$. Hence, by taking the supremum over all such ξ we obtain (2.10). $\square$

Corollary 2.1 *Let $0 < q < 2$ and $\|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1$. Then there exists a constant C depending on q only such that, for any $n \in \mathbb{N}$, there exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and weights $\omega_1, \dots, \omega_n$ such that*

$$\varrho_n(B_{X,\sigma}, L_2(U, X; \mu)) \leq \sup_{v \in B_{X,\sigma}} \|v - \tilde{S}_n^X v\|_{L_2(U, X; \mu)} \leq Cn^{-1/q},$$

where $\tilde{S}_n^X$ is defined as in (2.9).

Moreover,

$$(n+1)^{-1/q} \leq \sup_{\sigma: \|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1} \varrho_n(B_{X,\sigma}, L_2(U, X; \mu)) \asymp n^{-1/q}.$$

Proof. The upper bound in this theorem is derived from Lemmas 2.2 and 2.3, the asymptotic equivalence

$$\sqrt{\frac{1}{m} \sum_{k \geq m} k^{-2/q}} \asymp m^{-1/q}, \quad m \in \mathbb{N},$$

and from the fact that this bound is independent of the sequence σ . To prove the lower bound, one can take $\sigma = (\sigma_s)_{s \in \mathbb{N}}$ with $\sigma_s = (n+1)^{1/q}$ for $s \leq n+1$, and $\sigma_s = 2^{Ks/q}$ for $s > n+1$ with arbitrary $K \in \mathbb{N}$. It is easy to check that $\|\tilde{\sigma}^{-1}\|_{\ell_q(\mathbb{N})} \leq 1$, where $\tilde{\sigma} := (1 + 2^{-Kn})^{-1/q} \sigma$. We then have by Theorem 2.1 that

$$\begin{aligned} \varrho_n(B_{X,\tilde{\sigma}}, L_2(U, X; \mu)) &= \varrho_n(B_{\mathbb{C},\tilde{\sigma}}, L_2(U, \mathbb{C}; \mu)) \geq d_n(B_{\mathbb{C},\tilde{\sigma}}, L_2(U, \mathbb{C}; \mu)) \\ &= \tilde{\sigma}_{n+1}^{-1} = (n+1)^{-1/q} (1 + 2^{-Kn})^{-1/q}. \end{aligned}$$

Since K is arbitrary, we get

$$\sup_{\sigma: \|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1} \varrho_n(B_{X,\sigma}, L_2(U, X; \mu)) \geq n^{-1/q}.$$

□

Next, we apply Corollary 2.1 to Bochner spaces with infinite tensor-product probability measure, which appear in approximation of solutions to parametric PDEs with random inputs and of holomorphic functions in Sections 3 and 4, respectively.

For given $a, b > -1$, let $\nu_{a,b}$ be the Jacobi probability measure on $\mathbb{I} := [-1, 1]$ with the density

$$\delta_{a,b}(y) := c_{a,b}(1-y)^a(1+y)^b, \quad c_{a,b} := \frac{\Gamma(a+b+2)}{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)}.$$

Let $(J_k)_{k \in \mathbb{N}_0}$ be the sequence of Jacobi polynomials on $\mathbb{I} := [-1, 1]$ normalized with respect to the Jacobi probability measure $\nu_{a,b}$, i.e.,

$$\int_{\mathbb{I}} |J_k(y)|^2 d\nu_{a,b}(y) = \int_{\mathbb{I}} |J_k(y)|^2 \delta_{a,b}(y) dy = 1, \quad k \in \mathbb{N}_0.$$

Let γ be the standard Gaussian probability measure on $\mathbb{R}$ with the density

$$g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Let $(H_k)_{k \in \mathbb{N}_0}$ be the sequence of Hermite polynomials on $\mathbb{R}$ normalized with respect to the measure γ , i.e.,

$$\int_{\mathbb{R}} |H_k(y)|^2 d\gamma(y) = \int_{\mathbb{R}} |H_k(y)|^2 g(y) dy = 1, \quad k \in \mathbb{N}_0.$$

Throughout this section, we use the joint notation: $\mathbb{U}$ denotes either $\mathbb{I}$ or $\mathbb{R}$; $\mathbb{U}^\infty$ either $\mathbb{I}^\infty$ or $\mathbb{R}^\infty$;

$$\mu := \begin{cases} \nu_{a,b} & \text{if } \mathbb{U} = \mathbb{I}, \\ \gamma & \text{if } \mathbb{U} = \mathbb{R}; \end{cases}$$

$$\phi_k := \begin{cases} J_k & \text{if } \mathbb{U} = \mathbb{I}, \\ H_k & \text{if } \mathbb{U} = \mathbb{R}. \end{cases}$$

We next recall the concept of probability measure $\boldsymbol{\mu}(\mathbf{y})$ on $\mathbb{U}^\infty$ as the infinite tensor product of the measures $\mu(y_i)$:

$$\boldsymbol{\mu}(\mathbf{y}) := \bigotimes_{j \in \mathbb{N}} \mu(y_j), \quad \mathbf{y} = (y_j)_{j \in \mathbb{N}} \in \mathbb{U}^\infty.$$

(The sigma algebra for $\boldsymbol{\mu}(\mathbf{y})$ is generated by the set of cylinders $A := \prod_{j \in \mathbb{N}} A_j$, where $A_j \subset \mathbb{U}$ are univariate Lebesgue measurable sets and only a finite number of A_i are different from $\mathbb{U}$. For such a set A , we have $\boldsymbol{\mu}(A) = \prod_{j \in \mathbb{N}} \mu(A_j)$).

Let X be a separable Hilbert space. Then a function $v \in L_2(\mathbb{U}^\infty, X; \boldsymbol{\mu})$ can be represented by the generalized polynomial chaos (GPC) expansion

$$v = \sum_{\mathbf{s} \in \mathbb{F}} v_{\mathbf{s}} \phi_{\mathbf{s}}, \quad v_{\mathbf{s}} \in X, \quad (2.12)$$

with

$$\phi_{\mathbf{s}}(\mathbf{y}) = \bigotimes_{j \in \mathbb{N}} \phi_{s_j}(y_j), \quad v_{\mathbf{s}} := \int_{\mathbb{U}^\infty} v(\mathbf{y}) \phi_{\mathbf{s}}(\mathbf{y}) d\boldsymbol{\mu}(\mathbf{y}), \quad \mathbf{s} \in \mathbb{F}.$$

Here $\mathbb{F}$ is the set of all sequences of non-negative integers $\mathbf{s} = (s_j)_{j \in \mathbb{N}}$ such that their support $\text{supp}(\mathbf{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set.

For a set $\boldsymbol{\sigma} = (\sigma_{\mathbf{s}})_{\mathbf{s} \in \mathbb{F}}$ of positive numbers, denote by $B_{X, \boldsymbol{\sigma}}(\mathbb{U}^\infty)$ the set of all functions $v \in L_2(\mathbb{U}^\infty, X; \boldsymbol{\mu})$ represented by the series (2.12) such that

$$\left(\sum_{\mathbf{s} \in \mathbb{F}} (\sigma_{\mathbf{s}} \|v_{\mathbf{s}}\|_X)^2 \right)^{1/2} \leq 1.$$

Notice that if $\|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{F})} < \infty$ for some $0 < q \leq 2$, then for every $v \in B_{X, \boldsymbol{\sigma}}(\mathbb{U}^\infty)$, the series (2.12) converges absolutely and unconditionally in $L_2(\mathbb{U}^\infty, X; \boldsymbol{\mu})$ to v (see [24, Lemma 3.1] for the case $\mathbb{U}^\infty = \mathbb{R}^\infty$, the case $\mathbb{U}^\infty = \mathbb{I}^\infty$ can be proven by the same arguments). Hence, we can reorder the countable set $\mathbb{F}$ as $\mathbb{F} = (\mathbf{s}_j)_{j \in \mathbb{N}}$ so that the sequence $\boldsymbol{\sigma} = (\sigma_{\mathbf{s}_j})_{j \in \mathbb{N}}$ is non-decreasing. Put $U := \mathbb{U}^\infty$, $\sigma_j := \sigma_{\mathbf{s}_j}$, $\varphi_j := \phi_{\mathbf{s}_j}$ and $v_j := v_{\mathbf{s}_j}$. Then $B_{X, \boldsymbol{\sigma}}(\mathbb{U}^\infty)$ can be seen as the set $B_{X, \boldsymbol{\sigma}}$, defined as in Section 1, of all functions $v \in L_2(U, X; \boldsymbol{\mu})$ represented by the series

$$v = \sum_{j \in \mathbb{N}} v_j \varphi_j, \quad v_j \in X,$$

such that

$$\left(\sum_{j \in \mathbb{N}} (\sigma_j \|v_j\|_X)^2 \right)^{1/2} \leq 1.$$

In the next sections, due to this representation of $B_{X,\sigma}(\mathbb{U}^\infty)$, we are able to employ Corollary 2.1 in various applications.

3 Applications to parametric elliptic PDEs

3.1 Introductory remarks

For parametric PDEs, since the number of parametric variables may be very large or even infinite, they are treated as high-dimensional or infinite-dimensional approximation problems. As a model we consider parametric divergence-form elliptic PDEs with random inputs.

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Consider the diffusion elliptic equation

$$-\operatorname{div}(a(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in D, \quad u|_{\partial D} = 0, \quad (3.1)$$

for a given fixed right-hand side $f \in H^{-1}(D)$ and a spatially variable scalar diffusion coefficient a . Denote by $V := H_0^1(D)$ the energy space. If $a \in L_\infty(D)$ satisfies the ellipticity assumption

$$0 < a_{\min} \leq a \leq a_{\max} < \infty,$$

by the well-known Lax-Milgram lemma, there exists a weak unique solution $u \in V$ to the equation (3.1), satisfying

$$\int_D a(\mathbf{x}) \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, d\mathbf{x} = \langle f, v \rangle, \quad \forall v \in V.$$

PDEs with parametric and stochastic inputs are a common model used in science and engineering. Depending on the nature of the modeled object, the parameters involved in them may be either deterministic or random. The random nature reflects the uncertainty in various parameters presented in the physical phenomenon modeled by the equation. For equation (3.1), we consider the diffusion coefficients having a parametric form $a = a(\mathbf{y})$, where $\mathbf{y} = (y_j)_{j \in \mathbb{N}}$ is a sequence of real-valued parameters ranging in the set $\mathbb{U}^\infty$ which is either $\mathbb{R}^\infty$ or $\mathbb{I}^\infty$. Denote by $u(\mathbf{y})$ the solution to the parametric elliptic diffusion equation

$$-\operatorname{div}(a(\mathbf{y})(\mathbf{x})\nabla u(\mathbf{y})(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in D, \quad \mathbf{y} \in \mathbb{R}^\infty, \quad u(\mathbf{y})|_{\partial D} = 0, \quad \mathbf{y} \in \mathbb{U}^\infty. \quad (3.2)$$

The resulting solution operator maps $\mathbf{y} \in \mathbb{U}^\infty$ to $u(\mathbf{y}) \in V$. The objective is to achieve a numerical approximation of this complex map by a small number of parameters with a guaranteed error in a given norm.

In this section, we consider both the log-normal case when $\mathbb{U}^\infty = \mathbb{R}^\infty$ and the diffusion coefficient a is of the form

$$a(\mathbf{y}) = \exp(b(\mathbf{y})), \quad \text{with } b(\mathbf{y}) = \sum_{j=1}^{\infty} y_j \psi_j, \quad (3.3)$$

and y_j are i.i.d. standard Gaussian random variables, and the affine case when $\mathbb{U}^\infty = \mathbb{I}^\infty$ and the diffusion coefficient a is of the form

$$a(\mathbf{y}) = \bar{a} + \sum_{j=1}^{\infty} y_j \psi_j, \quad (3.4)$$

and y_j are i.i.d. standard Jacobi random variables. Here $\bar{a} \in L_\infty(D)$ and $\psi_j \in L_\infty(D)$ for both cases.

An approach to studying summability that takes into account the support properties of the component functions ψ_j , has been recently proposed in [9] for the affine parametric case, in [8] for the log-normal parametric case, and in [7] for extensions of both cases to second-order Sobolev norms of the corresponding GPC expansion coefficients. This approach leads to significant improvements on the results on ℓ_p -summability and weighted ℓ_2 -summability of GPC expansion coefficients, and therefore, on best n -term semi-discrete and fully discrete approximations when the component functions ψ_j have limited overlap, such as splines, finite elements or compactly supported wavelet bases. In this section, we will employ the results of the previous section to obtain convergence rates of sampling recovery of solutions to parametric elliptic PDEs with random inputs, which are derived from results on weighted ℓ_2 -summability in [7, 9].

3.2 Convergence rates

We first present some known weighted ℓ_2 -summability results for solutions u of parametric elliptic PDEs with random inputs.

For the log-normal case, we have the following result on weighted ℓ_2 -summability.

Lemma 3.1 *Let $0 < q < \infty$, $\eta \in \mathbb{N}$ with $\eta > 2/q$, and $\boldsymbol{\rho} := (\rho_j)_{j \in \mathbb{N}}$ be a sequence of positive numbers such that $\boldsymbol{\rho}^{-1} \in \ell_q(\mathbb{N})$, and for the log-normal parametric diffusion coefficient $a(\mathbf{y})$ as in (3.3),*

$$\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < \infty. \quad (3.5)$$

Then for the weak solution u to the parametric elliptic PDE (3.2) with the log-normal diffusion coefficient as in (3.3), there exist positive constants M, N such that

$$\left(\sum_{\mathbf{s} \in \mathbb{F}} (\sigma_{\mathbf{s}} \|u_{\mathbf{s}}\|_V)^2 \right)^{1/2} \leq M < \infty \quad \text{with} \quad \|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{F})} \leq N < \infty, \quad (3.6)$$

where with $|\mathbf{s}'|_\infty := \sup_{j \in \mathbb{N}} s'_j$ we define $\boldsymbol{\sigma} := \boldsymbol{\sigma}(\eta, \boldsymbol{\rho}) = (\sigma_{\mathbf{s}}(\eta, \boldsymbol{\rho}))_{\mathbf{s} \in \mathbb{F}}$ as

$$\sigma_{\mathbf{s}}^2 := \sigma_{\mathbf{s}}(\eta, \boldsymbol{\rho})^2 := \sum_{\mathbf{s}': |\mathbf{s}'|_\infty \leq \eta} \binom{\mathbf{s}}{\mathbf{s}'} \prod_{j \in \mathbb{N}} \rho_j^{2s'_j}, \quad \mathbf{s} \in \mathbb{F}. \quad (3.7)$$

Proof. A proof of this lemma is given in an implicit form in [8]. For completeness, let us present a short proof. By [8, Theorems 3.3 and 4.2], there exists a constant M such that

$$\sum_{\mathbf{s} \in \mathbb{F}} (\sigma_{\mathbf{s}} \|u_{\mathbf{s}}\|_V)^2 = \sum_{\mathbf{s}': |\mathbf{s}'|_\infty \leq \eta} \frac{\boldsymbol{\rho}^{2\mathbf{s}'}}{\mathbf{s}'!} \int_{\mathbb{R}^\infty} \left\| \partial^{\mathbf{s}'} u(\mathbf{y}) \right\|_V^2 d\gamma(\mathbf{y}) \leq M. \quad (3.8)$$

This proves the first inequality in (3.6). Since $\boldsymbol{\rho}^{-1} \in \ell_q(\mathbb{N})$, from [8, Lemma 5.1] the second inequality in (3.6) is implied. $\square$

For the affine case, we have the following result on weighted ℓ_2 -summability.

Lemma 3.2 *Let $\text{ess inf } \bar{a} > 0$. Let $0 < q < \infty$ and $(\rho_j)_{j \in \mathbb{N}}$ be a sequence of positive numbers such that $(\rho_j^{-1})_{j \in \mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$, and for the affine parametric diffusion coefficient $a(\mathbf{y})$ as in (3.4),*

$$\left\| \frac{\sum_{j \in \mathbb{N}} \rho_j |\psi_j|}{\bar{a}} \right\|_{L_\infty(D)} < 1. \quad (3.9)$$

Then for the weak solution u to the parametric elliptic PDE (3.2) with the affine diffusion coefficient as in (3.4), there exist positive constants M, N such that we have that

$$\left(\sum_{\mathbf{s} \in \mathbb{F}} (\sigma_{\mathbf{s}} \|u_{\mathbf{s}}\|_V)^2 \right)^{1/2} \leq M < \infty \quad \text{with} \quad \|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{F})} \leq N < \infty, \quad (3.10)$$

where $\boldsymbol{\sigma} := \boldsymbol{\sigma}(\boldsymbol{\rho}) = (\sigma_{\mathbf{s}}(\boldsymbol{\rho}))_{\mathbf{s} \in \mathbb{F}}$ is defined by

$$\sigma_{\mathbf{s}} := \sigma_{\mathbf{s}}(\boldsymbol{\rho}) := \prod_{j \in \mathbb{N}} c_{s_j}^{a,b} \rho_j^{s_j},$$

and $c_0^{a,b} := 1$,

$$c_k^{a,b} := \sqrt{\frac{(2k + a + b + 1)k! \Gamma(k + a + b + 1) \Gamma(a + 1) \Gamma(b + 1)}{\Gamma(k + a + 1) \Gamma(k + b + 1) \Gamma(a + b + 2)}}, \quad k \in \mathbb{N}.$$

Proof. A proof of this lemma is presented in an implicit form in [9] and [7]. The first inequality in (3.10) follows from [9, Remark 5.3], the second one from [7, (63)]. $\square$

Notice that assumptions (3.5) and (3.9) are different from the assumption $(\|\psi_j\|_{L_\infty(D)})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ considered in [13, 18, 19], or $(j\|\psi_j\|_{L_\infty(D)})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ considered in [31] for some $0 < p < 1$. The latter do not take into account the support properties of the component functions ψ_j , and hence, lead to worse results when the overlaps of the supports of ψ_j are finite. For a more detailed discussion on the advantages of assumptions (3.5) and (3.9) over these ones, we refer the reader to [9, 8, 22, 28].

We are now in a position to formulate the most significant result of our study. Let

$$V_m := \text{span}\{\phi_{\mathbf{s}_j}\}_{j=1}^m$$

be the subspace spanned by the (Hermite or Jacobi) polynomials $\phi_{\mathbf{s}_j}$, $j = 1, \dots, m$. Let $\mathbb{U}_0^\infty \subset \mathbb{U}^\infty$ be a fixed set of full measure such that the pointwise evaluations $u(\mathbf{y})$ are well-defined for every $\mathbf{y} \in \mathbb{U}_0^\infty$. (Such a set exists due to Lemma 3.1 or Lemma 3.2.) By applying Corollary 2.1, from Lemmata 3.1 and 3.2, and utilizing the homogeneous argument we obtain

Theorem 3.1 *Let the assumptions and notations of Lemma 3.1 or of Lemma 3.2 with $0 < q < 2$ hold for the log-normal case (3.3) ($\mathbb{U}^\infty = \mathbb{R}^\infty$) or for the affine case (3.4) ($\mathbb{U}^\infty = \mathbb{I}^\infty$), respectively. Let u be the weak solution to the parametric elliptic PDE (3.2) with the log-normal diffusion coefficient as in (3.3) or the affine diffusion coefficient as in (3.4), respectively. Then for any $n \in \mathbb{N}$, there exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{U}_0^\infty$ and weights $\omega_1, \dots, \omega_n$ such that*

$$\left\| u - \tilde{S}_n^V u \right\|_{L_2(\mathbb{U}^\infty, V; \mu)} \leq CMNn^{-1/q}$$

with a constant C independent of n, M, N and u , where $\tilde{S}_n^V$ is defined as in (2.9) for $X = V$.

4 Applications to holomorphic functions

The sparsity analysis for parametric elliptic PDEs with log-normal diffusion coefficients, as in [7, 8], hinges on real-variable bootstrapping to establish sparsity. This approach encounters technical obstacles when extending to higher spatial regularity or more general parametric PDEs. By contrast, complex-variable methods proposed in [28], using holomorphic extension of the solution offer an alternative pathway to sparsity and regularity, often simplifying the treatment of smoothness and broadening applicability. One advantage of establishing sparsity of Hermite GPC expansion coefficients via holomorphy rather than by successive differentiation is that it allows to derive, in a unified way, weighted ℓ_2 -summability bounds for the coefficients of Hermite GPC expansion whose size is measured in Sobolev scales in the domain D .

Formally, in the log-normal case (3.3) of the parametric equation (3.2), replacing $\mathbf{y} = (y_j)_{j \in \mathbb{N}} \in \mathbb{R}^\infty$ in the coefficient $a(\mathbf{y})$ in (3.3) by $\mathbf{z} = (z_j)_{j \in \mathbb{N}} = (y_j + i\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^\infty$, the real part of $a(\mathbf{z})$ is

$$\Re[a(\mathbf{z})] = \exp \left(\sum_{j \in \mathbb{N}} y_j \psi_j(\mathbf{x}) \right) \cos \left(\sum_{j \in \mathbb{N}} \xi_j \psi_j(\mathbf{x}) \right).$$

We find that $\Re[a(\mathbf{z})] > 0$ if

$$\left\| \sum_{j \in \mathbb{N}} \xi_j \psi_j \right\|_{L_\infty(D)} < \frac{\pi}{2}.$$

This motivates the study of the analytic continuation of the solution map $\mathbf{y} \mapsto u(\mathbf{y})$ to $\mathbf{z} \mapsto u(\mathbf{z})$ for complex parameters $\mathbf{z} = (z_j)_{j \in \mathbb{N}}$ where each z_j lies in the strip

$$\mathcal{S}_j(\boldsymbol{\rho}) := \{z_j \in \mathbb{C} : |\Im z_j| < \rho_j\} \quad (4.1)$$

and where $\rho_j > 0$ and $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}}$ is any sequence of positive numbers such that

$$\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < \frac{\pi}{2}.$$

Let $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}}$ be a sequence of non-negative numbers and assume that $J \subseteq \text{supp}(\boldsymbol{\rho})$ is finite. Define for $\mathbf{y} \in \mathbb{R}^\infty$,

$$\mathcal{S}_J(\mathbf{y}, \boldsymbol{\rho}) := \{(z_j)_{j \in \mathbb{N}} \in \mathbb{C}^\infty : z_j \in \mathcal{S}_j(\boldsymbol{\rho}) \text{ if } j \in J, \text{ and } z_j = y_j \text{ if } j \notin J\}. \quad (4.2)$$

For the definition of Sobolev spaces $H^r := H^r(D)$ and $W_\infty^r := W_\infty^r(D)$ as well as that of a C^m -domain see, e.g., [1]. The following result on holomorphy of the parametric solution has been proven in [28, Proposition 3.21].

Lemma 4.1 *Let $r \in \mathbb{N}$ and D be a bounded domain with either C^∞ -boundary or convex C^{r-1} -boundary. Let the sequence $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}} \in [0, \infty)^\infty$ satisfy*

$$\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} \leq \kappa < \frac{\pi}{2}. \quad (4.3)$$

Let $\mathbf{y}_0 = (y_{0,1}, y_{0,2}, \dots) \in \mathbb{R}^\infty$ be such that $b(\mathbf{y}_0)$ belongs to W_∞^{r-1} , and let $J \subseteq \text{supp}(\boldsymbol{\rho})$ be a finite set. Then the weak parametric solution u of the variational form of (3.2) with log-lognormal random inputs (3.3) is holomorphic on $\mathcal{S}_J(\mathbf{y}_0, \boldsymbol{\rho})$ as a function of the parameters $\mathbf{z}_J = (z_j)_{j \in J} \in \mathcal{S}_J(\mathbf{y}_0, \boldsymbol{\rho})$ taking values in $H^r(D)$ with $z_j = y_{0,j}$ for $j \notin J$ held fixed.

Based on the holomorphy of the parametric solution as in Lemma 4.1, a weighted ℓ_2 -summability of the Sobolev H^r -norm of the Hermite GPC expansion coefficients of the parametric solution u has been established in [28, Theorem 3.25] as follows.

Lemma 4.2 *Let $r \in \mathbb{N}$, D be a bounded domain with either C^∞ -boundary or convex C^{r-1} -boundary, and $f \in H^{r-1}(D)$. Assume that for every $j \in \mathbb{N}$, $\psi_j \in W_\infty^{r-1}$, and there exists a positive sequence $(\lambda_j)_{j \in \mathbb{N}}$ such that $(\exp(-\lambda_j^2))_{j \in \mathbb{N}} \in \ell_1(\mathbb{N})$ and the series $\sum_{j \in \mathbb{N}} \lambda_j |D^\alpha \psi_j|$ converges in $L_\infty(D)$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq r-1$. Let $\boldsymbol{\varrho} = (\varrho_j)_{j \in \mathbb{N}}$ be a sequence of positive numbers satisfying $(\varrho_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$ for some $0 < q < \infty$. Assume that, for each $\mathbf{s} \in \mathbb{F}$, there exists a sequence $\boldsymbol{\rho}_\mathbf{s} = (\rho_{\mathbf{s},j})_{j \in \mathbb{N}}$ of non-negative numbers such that $\text{supp}(\mathbf{s}) \subseteq \text{supp}(\boldsymbol{\rho}_\mathbf{s})$,*

$$\sup_{\mathbf{s} \in \mathbb{F}} \sum_{|\alpha| \leq s-1} \left\| \sum_{j \in \mathbb{N}} \rho_{\mathbf{s},j} |D^\alpha \psi_j| \right\|_{L_\infty(D)} \leq \kappa < \frac{\pi}{2}, \quad \text{and} \quad \sum_{\mathbf{s}': |\mathbf{s}'|_\infty \leq \eta} \frac{s! \varrho^{2\mathbf{s}}}{\rho_{\mathbf{s}}^{2\mathbf{s}}} < \infty \quad (4.4)$$

with $\eta \in \mathbb{N}$, $\eta > 2/q$. Let u be the weak solution to the parametric elliptic PDE (3.2) with the log-normal diffusion coefficient as in (3.3). Then exist positive constants M, N such that

$$\sum_{\mathbf{s} \in \mathbb{F}} (\sigma_\mathbf{s}(\eta, \boldsymbol{\varrho}) \|u_\mathbf{s}\|_{H^r})^2 \leq M < \infty \quad \text{with} \quad \|\boldsymbol{\sigma}(\eta, \boldsymbol{\varrho})^{-1}\|_{\ell_q(\mathbb{F})} \leq N < \infty, \quad (4.5)$$

where $\boldsymbol{\sigma}(\eta, \boldsymbol{\varrho}) = (\sigma_\mathbf{s}(\eta, \boldsymbol{\varrho}))_{\mathbf{s} \in \mathbb{F}}$ is given by (3.7).

By applying Corollary 2.1, from Lemma 4.2, under the assumptions and notation of Lemma 4.2, and utilizing the homogeneous argument we again obtain

Corollary 4.1 *Let the assumptions and notation of Lemma 4.2 hold for some $0 < q < 2$. Let u be the weak solution to the parametric elliptic PDE (3.2) with the log-normal diffusion coefficient as in (3.3). Then for any $n \in \mathbb{N}$, there exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{U}_0^\infty$ and weights $\omega_1, \dots, \omega_n$ such that*

$$\left\| u - \tilde{S}_n^{H^r} u \right\|_{L_2(\mathbb{R}^\infty, H^r; \boldsymbol{\gamma})} \leq CMN n^{-1/q} \quad (4.6)$$

with a constant C independent of n, M, N and u , where $\tilde{S}_n^{H^r}$ is defined as in (2.9) for $X = H^r$.

The convergence rate in (4.6) of Corollary 4.1 is an entirely new result for semi-discrete non-intrusive approximation of the parametric solution $u(\mathbf{y})$, which significantly improved by a factor $n^{-1/2}$ the previous result which is implied from [28, Theorem 6.13] for $X = H^r$.

The results of Lemmata 4.1 and 4.2 encourage us to investigate the holomorphy and weighted ℓ_2 -summability as a sequence for a wider class of functions on $\mathbb{R}^\infty$ and application to approximation for parametric PDEs with log-normal random inputs. We recall the concept of “ $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic functions” on $\mathbb{R}^\infty$ which has been introduced in [28, Definition 4.1] for general parametric PDEs with random input data. For $m \in \mathbb{N}$ and a positive sequence $\boldsymbol{\varrho} = (\varrho_j)_{j=1}^m$, we put

$$\mathcal{S}(\boldsymbol{\varrho}) := \{\mathbf{z} \in \mathbb{C}^m : |\Im z_j| < \varrho_j \ \forall j\} \quad \text{and} \quad \mathcal{B}(\boldsymbol{\varrho}) := \{\mathbf{z} \in \mathbb{C}^m : |z_j| < \varrho_j \ \forall j\}.$$

Let X be a complex separable Hilbert space, $\mathbf{b} = (b_j)_{j \in \mathbb{N}}$ a positive sequence, and $\xi > 0$, $\varepsilon > 0$. For $m \in \mathbb{N}$ we say that a positive sequence $\boldsymbol{\varrho} = (\varrho_j)_{j=1}^m$ is $(\mathbf{b}, \xi)$ -admissible if

$$\sum_{j=1}^m b_j \varrho_j \leq \xi.$$

A function $v \in L_2(\mathbb{R}^\infty, X; \gamma)$ is called $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic if

- (i) for every $m \in \mathbb{N}$ there exists $v_m : \mathbb{R}^m \rightarrow X$, which, for every $(\mathbf{b}, \xi)$ -admissible $\boldsymbol{\varrho}$, admits a holomorphic extension (denoted again by v_m) from $\mathcal{S}(\boldsymbol{\varrho}) \rightarrow X$; furthermore, for all $m < m'$

$$v_m(y_1, \dots, y_m) = v_{m'}(y_1, \dots, y_m, 0, \dots, 0) \quad \forall (y_j)_{j=1}^m \in \mathbb{R}^m,$$

- (ii) for every $m \in \mathbb{N}$ there exists $\varphi_m : \mathbb{R}^m \rightarrow \mathbb{R}_+$ such that $\|\varphi_m\|_{L_2(\mathbb{R}^m; \gamma)} \leq \varepsilon$ and

$$\sup_{\boldsymbol{\varrho} \text{ is } (\mathbf{b}, \xi)\text{-adm.}} \sup_{\mathbf{z} \in \mathcal{B}(\boldsymbol{\varrho})} \|v_m(\mathbf{y} + \mathbf{z})\|_X \leq \varphi_m(\mathbf{y}) \quad \forall \mathbf{y} \in \mathbb{R}^m,$$

- (iii) with $\tilde{v}_m : \mathbb{R}^\infty \rightarrow X$ defined by $\tilde{v}_m(\mathbf{y}) := v_m(y_1, \dots, y_m)$ for $\mathbf{y} \in \mathbb{R}^\infty$ it holds

$$\lim_{m \rightarrow \infty} \|v - \tilde{v}_m\|_{L_2(\mathbb{R}^\infty, X; \gamma)} = 0.$$

We notice some important examples of $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic functions on $\mathbb{R}^\infty$ which are solutions to parametric PDEs equations with log-normal random inputs and which were studied in [28]. Let $b(\mathbf{y})$ be defined as in (3.3) and $\mathcal{V}$ a holomorphic map from an open set in $L_\infty(D)$ to X . Then function compositions of the type

$$v(\mathbf{y}) = \mathcal{V}(\exp(b(\mathbf{y})))$$

are $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic under certain conditions [28, Proposition 4.11]. This allows us to apply weighted ℓ_2 -summability for approximation of solutions $v(\mathbf{y}) = \mathcal{V}(\exp(b(\mathbf{y})))$ as $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic functions on various function spaces X , to a wide range of parametric and stochastic PDEs with log-normal inputs. Such function spaces X are high-order regularity spaces $H^s(D)$ [28, Section 4.3.1] and corner-weighted Sobolev (Kondrat'ev) spaces $K_\varkappa^s(D)$ ($s \geq 1$) for the parametric elliptic PDEs (3.1) with log-normal inputs (3.3) [28, Section 7.6.1]; spaces of solutions to linear

parabolic PDEs with log-normal inputs (3.3) [28, Section 4.3.2]; spaces of solutions to linear elastic equations with log-normal modulus of elasticity [28, Section 4.3.3]; spaces of solutions to Maxwell equations with log-normal permittivity [28, Section 4.3.4]; spaces of posterior densities and of their linear functionals in Bayesian inverse problems [28, Section 5], etc..

The following key result on weighted ℓ_2 -summability of $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic functions has been proven in [28, Theorem 4.9].

Lemma 4.3 *Let v be $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic for some $\mathbf{b} \in \ell_p(\mathbb{N})$ with $0 < p < 1$. Let $\eta \in \mathbb{N}$ and let the sequence $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}}$ be defined by*

$$\rho_j := b_j^{p-1} \frac{\xi}{4\sqrt{\eta!} \|\mathbf{b}\|_{\ell_p(\mathbb{N})}}.$$

Then we have

$$\left(\sum_{\mathbf{s} \in \mathbb{F}} (\sigma_{\mathbf{s}} \|v_{\mathbf{s}}\|_X)^2 \right)^{1/2} \leq M < \infty, \quad \text{with} \quad \|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{F})} \leq N < \infty,$$

where $q := p/(1-p)$, $\boldsymbol{\sigma} := \boldsymbol{\sigma}(\eta, \boldsymbol{\rho}) = (\sigma_{\mathbf{s}}(\eta, \boldsymbol{\rho}))_{\mathbf{s} \in \mathbb{F}}$ is given by (3.7), $M = \varepsilon C_{\mathbf{b}}$ and $N = C_{\mathbf{b}, \xi}$ with some positive constants $C_{\mathbf{b}}$ and $C_{\mathbf{b}, \xi}$.

Let $\mathbb{R}_0^\infty \subset \mathbb{R}^\infty$ be a fixed set of full Gaussian measure such that the pointwise evaluations $u(\mathbf{y})$ are well-defined for every $\mathbf{y} \in \mathbb{R}_0^\infty$. (Such a set exists due to Lemma 4.3.) By applying Corollary 2.1, from Lemma 4.3, and utilizing the homogeneous argument we obtain

Theorem 4.1 *Let v be $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic for some $\mathbf{b} \in \ell_p(\mathbb{N})$ with $0 < p < 2/3$. Then for any $n \geq 2$, there exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{R}_0^\infty$ and weights $\omega_1, \dots, \omega_n$ such that*

$$\left\| v - \tilde{S}_n^X v \right\|_{L_2(\mathbb{R}^\infty, X; \gamma)} \leq CMN n^{-(1/p-1)} \quad (4.7)$$

with a constant C independent of n and v , where M, N are as in Lemma 4.3 and $\tilde{S}_n^X$ is defined as in (2.9).

The convergence rate in Theorem 4.1 notably improves the result [28, Theorem 6.13] by a factor $n^{-1/2}$.

We present two examples of application of Theorem 4.1 to parametric PDEs with random inputs.

Let us revisit the parametric equation (3.2) with log-normal random inputs (3.3). Hence, if $(\psi_j)_{j \in \mathbb{N}} \subset L_\infty(D)$ such that $\mathbf{b} \in \ell_1(\mathbb{N})$ with $b_j := \|\psi_j\|_{L_\infty(D)}$, then the parametric weak solution $u(\mathbf{y})$ uniquely exists and is $(\mathbf{b}, \xi, \varepsilon, V)$ -holomorphic by [28, Theorem 4.11]. For details, see [28, Section 4.3.1].

Assume in addition, that $\mathbf{b}$ is a sequence such that $\mathbf{b} \in \ell_p(\mathbb{N})$ for some $0 < p < 2/3$. Then the parametric solution $u(\mathbf{y})$ can be approximated by the extended least squares sampling algorithm $\tilde{S}_n^V$ defined as in Theorem 4.1 for $X = V$, for which it holds the convergence rate (4.7).

We apply Theorem 4.1 to solutions to parametric linear parabolic PDEs with log-normal random inputs. To establish this, let us present the holomorphic properties of solutions of these

equations. For details, see [28, Section 4.3.2]. Let $0 < T < \infty$ denote a finite time-horizon and let D be a bounded domain with Lipschitz boundary ∂D in $\mathbb{R}^d$. We define $I := (0, T)$ and consider the initial boundary value problem for the linear parabolic PDE

$$\begin{cases} \frac{\partial u(t, \mathbf{x})}{\partial t} - \operatorname{div}(a(\mathbf{x}) \nabla u(t, \mathbf{x})) = f(t, \mathbf{x}), & (t, \mathbf{x}) \in I \times D, \\ u|_{\partial D \times I} = 0, \\ u|_{t=0} = u_0(\mathbf{x}). \end{cases} \quad (4.8)$$

We denote $V := H_0^1(D; \mathbb{C})$ and $V' := H^{-1}(D; \mathbb{C})$. Let

$$X := L_2(I, V) \cap H^1(I, V') = (L_2(I) \otimes V) \cap (H^1(I) \otimes V') \quad (4.9)$$

equipped with the sum norm

$$\|u\|_X := \left(\|u\|_{L_2(I, V)}^2 + \|u\|_{H^1(I, V')}^2 \right)^{1/2}, \quad u \in X, \quad (4.10)$$

where

$$\|u\|_{L_2(I, V)}^2 = \int_I \|u(t, \cdot)\|_V^2 dt,$$

and

$$\|u\|_{H^1(I, V')}^2 = \int_I \|\partial_t u(t, \cdot)\|_{V'}^2 dt.$$

To state a space-time variational formulation and to specify the data space for (4.8), we introduce the test-function space

$$Y = L_2(I, V) \times L_2(D) = (L_2(I) \otimes V) \times L_2(D)$$

which we endow with the norm

$$\|v\|_Y = \left(\|v_1\|_{L_2(I, V)}^2 + \|v_2\|_{L_2(D)}^2 \right)^{1/2}, \quad v = (v_1, v_2) \in Y.$$

Given a time-independent diffusion coefficient $a \in L_\infty(D; \mathbb{C})$ and $(f, u_0) \in Y'$, the continuous sesqui-linear and anti-linear forms corresponding to the parabolic problem (4.8) reads for $u \in X$ and $v = (v_1, v_2) \in Y$ as

$$B(u, v) := \int_I \int_D \partial_t u \overline{v_1} d\mathbf{x} dt + \int_I \int_D a \nabla u \cdot \overline{\nabla v_1} d\mathbf{x} dt + \int_D u_0 \overline{v_2} d\mathbf{x}$$

and

$$L(v) := \int_I \langle f(t, \cdot), v_1(t, \cdot) \rangle dt + \int_D u_0 \overline{v_2} d\mathbf{x},$$

where $\langle \cdot, \cdot \rangle$ is the anti-duality pairing between V' and V . Then the space-time variational (weak) formulation of equation (4.8) is: Find a weak solution $u \in X$ such that

$$B(u, v) = L(v), \quad \forall v \in Y. \quad (4.11)$$

Assume that $(f, u_0) \in Y'$ and that

$$0 < \rho(a) := \operatorname{ess\,inf}_{\mathbf{x} \in D} \Re(a(\mathbf{x})) \leq |a(\mathbf{x})| \leq \|a\|_{L_\infty(D)} < \infty, \quad \mathbf{x} \in D. \quad (4.12)$$

Then there exists a unique solution u to the equation (4.11).

Consider the initial boundary value problem for the parametric linear parabolic PDE

$$\begin{cases} \frac{\partial u(\mathbf{y})(t, \mathbf{x})}{\partial t} - \operatorname{div}(a(\mathbf{y})(\mathbf{x}) \nabla u(\mathbf{y})(t, \mathbf{x})) = f(t, \mathbf{x}), & (t, \mathbf{x}) \in I \times D, \quad \mathbf{y} \in \mathbb{R}^\infty, \\ u(\mathbf{y})(\mathbf{x})|_{\partial D \times I} = 0, & \mathbf{y} \in \mathbb{R}^\infty, \\ u(\mathbf{y})(\mathbf{x})|_{t=0} = u_0(\mathbf{x}), & \mathbf{y} \in \mathbb{R}^\infty, \end{cases} \quad (4.13)$$

with log-normal inputs (3.3). Hence, if $(\psi_j)_{j \in \mathbb{N}} \subset D$ such that $\mathbf{b} \in \ell_1(\mathbb{N})$ with $b_j := \|\psi_j\|_{L_\infty(D)}$, then the parametric weak solution $u(\mathbf{y})$ uniquely exists and $(\mathbf{b}, \xi, \varepsilon, X)$ -holomorphic by [28, Theorem 4.11].

Assume in addition, that $\mathbf{b}$ is a sequence such that $\mathbf{b} \in \ell_p(\mathbb{N})$ for some $0 < p < 2/3$. Then the parametric weak solution $u(\mathbf{y})$ to the equation (4.13) can be approximated by the extended least squares sampling algorithm $\tilde{S}_n^X$ defined in Theorem 4.1 for the space X as in (4.9) and (4.10), for which holds the convergence rate (4.7).

5 Constructiveness and alternative least squares methods

In this section, we present several different sampling schemes to use with the least squares algorithms for functions in the RKHS $H_{\mathbb{C}, \sigma}$, and inequalities between sampling n -widths and Kolmogorov n -widths of the unit ball $B_{\mathbb{C}, \sigma}$. We explain then how to apply these inequalities to obtain corresponding convergence rates of linear sampling recovery in abstract Bochner spaces and of approximation of the parametric solution $u(\mathbf{y})$ to parametric elliptic or parabolic PDEs with log-normal inputs as well as of infinite-dimensional holomorphic functions.

The choice of points $\mathbf{y}_1, \dots, \mathbf{y}_n$, weights $\omega_1, \dots, \omega_n$, and approximation space V_m is crucial for the error of the least squares approximation. A lot of work has been done in the usual Lebesgue space $L_2(U, \mathbb{C}; \mu)$ of which we present two more choices with a trade-off between constructiveness and tightness of the bound and transfer them to the Bochner space $L_2(U, X; \mu)$.

For $m \in \mathbb{N}$, let the probability measure $\nu = \nu(m)$ introduced in [33], be defined by

$$d\nu(\mathbf{y}) := \varrho(\mathbf{y}) d\mu(\mathbf{y}) := \frac{1}{2} \left(\frac{1}{m} \sum_{s=1}^m |\varphi_s(\mathbf{y})|^2 + \frac{\sum_{s=m+1}^\infty |\sigma_s^{-1} \varphi_s(\mathbf{y})|^2}{\sum_{s=m+1}^\infty \sigma_s^{-2}} \right) d\mu(\mathbf{y}). \quad (5.1)$$

Assumption 5.1 *Let $m \geq 3$.*

- (i) *Let $n = \lceil 40m \log m \rceil$. Let further $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ be points drawn i.i.d. with respect to ν and $\omega_i := (\varrho(\mathbf{y}_i))^{-1}$.*
- (ii) *Let $m := n$ and $\lceil 40n \log n \rceil$ points be drawn i.i.d. with respect to ν and subsampled using [11, Algorithm 3] to $n = \lceil bm \rceil$ points for some $b > 1 + 1/m$. Denote the resulting points by $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and $\omega_i = \frac{n}{\lceil 40n \log n \rceil} (\varrho(\mathbf{y}_i))^{-1}$.*

Note that (i) and (ii) in Assumption 5.1 have been considered in [33] and [11].

Recall that we use the abbreviation $d_n := d_n(B_{\mathbb{C}, \sigma}, L_2(U, \mathbb{C}; \mu))$.

Lemma 5.1 *Let $S_n^X := S_n^X(\mathbf{y}_1, \dots, \mathbf{y}_n, \omega_1, \dots, \omega_n, V_m)$ be the extended least squares algorithm defined as in (2.8). Then we have the following.*

(i) *The points from Assumption 5.1(i) fulfill with probability exceeding $1 - (160 \log n)/n$*

$$\sup_{v \in B_{X, \sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} \leq 19 \max \left\{ d_{\lfloor n/(40 \log n) \rfloor}, \sqrt{\frac{\log n}{n} \sum_{s \geq \lfloor n/(40 \log n) \rfloor} d_s^2} \right\}.$$

(ii) *The points from Assumption 5.1(ii) fulfill with probability exceeding $1 - 8b/n$*

$$\sup_{v \in B_{X, \sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} \leq 552 \left(\frac{b+1}{b-1} \right)^{3/2} \max \left\{ \sqrt{\log n} d_{\lfloor n/(2b) \rfloor}, \sqrt{\frac{\log n}{n} \sum_{s \geq \lfloor n/(2b) \rfloor} d_s^2} \right\}.$$

(iii) *There exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and weights $\omega_i \geq 0$ such that*

$$\sup_{v \in B_{X, \sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} \leq 4325 \max \left\{ d_{\lfloor n/43200 \rfloor}, \sqrt{\frac{1}{n} \sum_{s \geq \lfloor n/43200 \rfloor} d_s^2} \right\}.$$

Proof. by Theorem 2.1, it suffices to show the inequalities $X = \mathbb{C}$. To prove (i) we use [10, Theorem 7.7] with $t = \log m$. An intermediate step in the proof states with probability exceeding $1 - 4 \exp(-t)$

$$\sup_{v \in B_{X, \sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} \leq \sqrt{3d_m^2 + \frac{168 \log n}{n} \sum_{s \geq m+1} d_s^2} \leq 19 \max \left\{ d_m, \sqrt{\frac{\log n}{n} \sum_{s \geq m+1} d_s^2} \right\}.$$

From the assumption on n we have $n/(40 \log n) \leq m$, which yields the first assertion and the desired probability $1 - \exp(-t) = 1 - 4/m \geq 1 - (160 \log n)/n$.

To prove (ii) we use [10, Theorem 7.8] with $t = \log m$. An intermediate step in the proof states with probability exceeding $1 - 4 \exp(-t) = 1 - 4/m \leq 1 - 8b/n$

$$\sup_{v \in B_{X, \sigma}} \|v - S_n^X v\|_{L_2(U, X; \mu)} \leq \sqrt{\frac{10858}{3} \frac{(b+1)^2}{(b-1)^3} \log m \left(2d_m^2 + \frac{168 \log \lceil 40m \log m \rceil}{\lceil 40m \log m \rceil} \sum_{s \geq m+1} d_s^2 \right)}.$$

With $m \geq 3$ we have

$$\frac{168 \log \lceil 40m \log m \rceil}{\lceil 40m \log m \rceil} \leq \frac{168 \log(41m \log m)}{40m \log m} \leq \frac{21}{m}.$$

Thus

$$\begin{aligned} \sup_{v \in B_{X,\sigma}} \|v - S_n^X v\|_{L_2(U,X;\mu)} &\leq \sqrt{\frac{10858}{3} \frac{(b+1)^2}{(b-1)^3} \log m \left(2d_m^2 + \frac{21}{m} \sum_{s \geq m+1} d_s^2 \right)} \\ &\leq \sqrt{304304 \left(\frac{b+1}{b-1} \right)^3 \log n \max \left\{ d_{\lfloor n/(2b) \rfloor}^2, \frac{1}{n} \sum_{s \geq \lfloor n/(2b) \rfloor} d_s^2 \right\}}. \end{aligned}$$

Part (iii) of the assertion is given in [27, Theorem 23] (see Lemma 2.2). $\square$

Regarding the constructiveness of the linear sampling algorithms in Lemma 5.1, the bound Lemma 5.1(i) is the most coarse bound, but the points construction requires only a random draw, which is computationally inexpensive. The sharper bound in Lemma 5.1(ii) uses an additional constructive subsampling step. This was implemented and numerically tested in [11] for up to 1000 basis functions. For larger problem sizes the current algorithm is too slow as its runtime is cubic in the number of basis functions. The sharpest bound in Lemma 5.1(iii) is a pure existence result. So, the only way to obtain this point set is to brute-force every combination, which is computational infeasible.

Similarly to the proof of Corollary 2.1, one can prove the following results on convergence rates of the extended least squares sampling algorithms described in Lemma 5.1.

Corollary 5.1 *Let $0 < q < 2$ and $\|\sigma^{-1}\|_{\ell_q(\mathbb{N})} \leq 1$. Let $S_n^X := S_n^X(\mathbf{y}_1, \dots, \mathbf{y}_n, \omega_1, \dots, \omega_n, V_m)$ be the extended least squares algorithm defined as in (2.8). There are universal constants $c_1, c_2, c_3 \in \mathbb{N}$ such that for all $n \geq 2$ we have the following.*

(i) *The points from Assumption 5.1(i) fulfill with high probability*

$$\sup_{v \in B_{X,\sigma}} \|v - S_{c_1 n}^X v\|_{L_2(U,X;\mu)} \lesssim n^{-1/q} (\log n)^{1/q};$$

(ii) *The points from Assumption 5.1(ii) fulfill with high probability*

$$\sup_{v \in B_{X,\sigma}} \|v - S_{c_2 n}^X v\|_{L_2(U,X;\mu)} \lesssim n^{-1/q} (\log n)^{1/2}.$$

(iii) *There exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and weights $\omega_i \geq 0$ such that*

$$\sup_{v \in B_{X,\sigma}} \|v - S_{c_3 n}^X v\|_{L_2(U,X;\mu)} \lesssim n^{-1/q}.$$

One can apply the last corollary to parametric PDEs with random inputs and infinite-dimensional holomorphic functions to receive counterparts of all the results in Sections 3 and 4 on extended least squares sampling recovery, based on the sample points from Assumption 5.1(i)–(iii). In particular, by applying Corollary 5.1, from Lemmata 3.1 and 3.2 we obtain

Corollary 5.2 *Let $S_n^V := S_n^V(\mathbf{y}_1, \dots, \mathbf{y}_n, \omega_1, \dots, \omega_n, V_m)$ be the extended least squares algorithm defined as in (2.8) for $X = V$. Under the assumption of Theorem 3.1, there are universal constants $c_1, c_2, c_3 \in \mathbb{N}$ such that for all $n \geq 2$ we have the following.*

(i) The points from Assumption 5.1(i) fulfill with high probability

$$\|u - S_{c_1 n}^V u\|_{L_2(\mathbb{U}^\infty, V; \boldsymbol{\mu})} \leq CMNn^{-1/q}(\log n)^{1/q};$$

(ii) The points from Assumption 5.1(ii) fulfill with high probability

$$\|u - S_{c_2 n}^V u\|_{L_2(\mathbb{U}^\infty, V; \boldsymbol{\mu})} \leq CMNn^{-1/q}(\log n)^{1/2};$$

(iii) There exist points $\mathbf{y}_1, \dots, \mathbf{y}_n \in U_0$ and weights $\omega_i \geq 0$ such that

$$\|u - S_{c_3 n}^V u\|_{L_2(\mathbb{U}^\infty, V; \boldsymbol{\mu})} \leq CMNn^{-1/q};$$

The constants C in the above inequalities are independent of M, N, n and u .

In the affine case (3.4), the convergence rate $(n/\log n)^{-1/q}$ (with $1/q := 1/p - 1/2$) in terms of the number n of sampling points has been received in [14] for an *adaptive* least squares approximation “lifting” to Hilbert-valued functions, based on an ℓ_q -summability of the Legendre GPC expansion coefficients of the parametric solution, and on an adaptive choice of sequence of finite dimensional approximation spaces, which is different from the linear extended least squares approximation in Corollary 5.2(i). Notice also that the result in Corollary 5.2(i) for the affine case could be also proven by a linear modification of the technique used in [14], based on the weighted ℓ_2 -summability (3.10). The convergence rate $(n/\log n)^{-1/q}$ (with $1/q := 1/p - 1/2$) of sampling recovery of $(\mathbf{b}, \varepsilon)$ -holomorphic functions on $\mathbb{I}^\infty$ has been proven in [5] based on least squares procedure of [34] “lifting” to Hilbert-valued functions, where $\varepsilon > 0$, $\mathbf{b} = (b_j)_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ is a positive sequence and $0 < p < 1$.

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